University of East Anglia

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Controlling the complex dynamics of multilayer shear flows using surfactants

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In collaboration with:

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Outline

1. Introduction
2. Mathematical model: derivation and validation
3. The effect of surfactants
   - Linear stability theory
   - Nonlinear solutions
   - Destabilisation mechanism
   - Complex dynamics: quasi-periodicity and chaos
4. Extension of three dimensions
5. Summary and future work

Surfactant: **Surf**ace **act**ing **ag**e**nt**

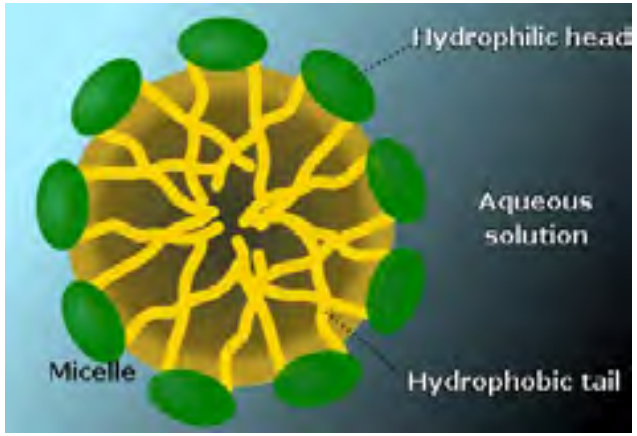
- Detergents & soaps
- Bipolar molecules: hydrophilic head, hydrophobic tail

Equation of state

$$\gamma = 1 - \beta \Gamma$$

Effect of surfactant:

- Reduces the surface tension
- Increases the surface area



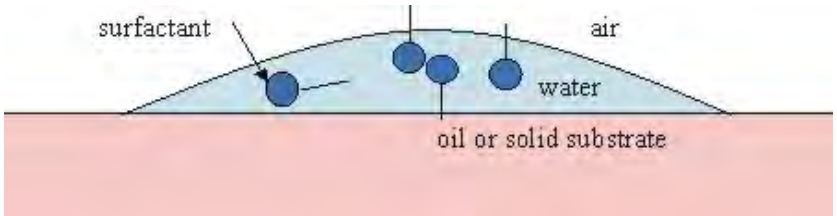
Marangoni effect

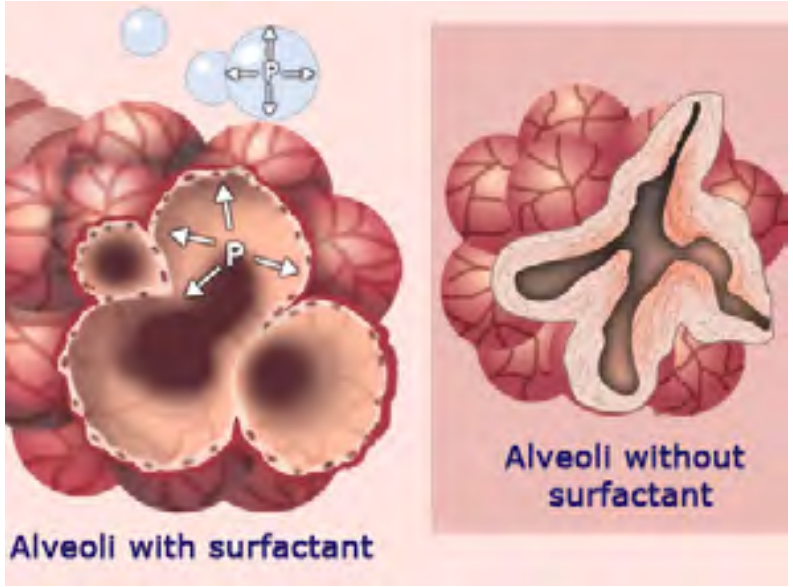
The mass transfer along an interface between two fluids due to surface tension gradients.

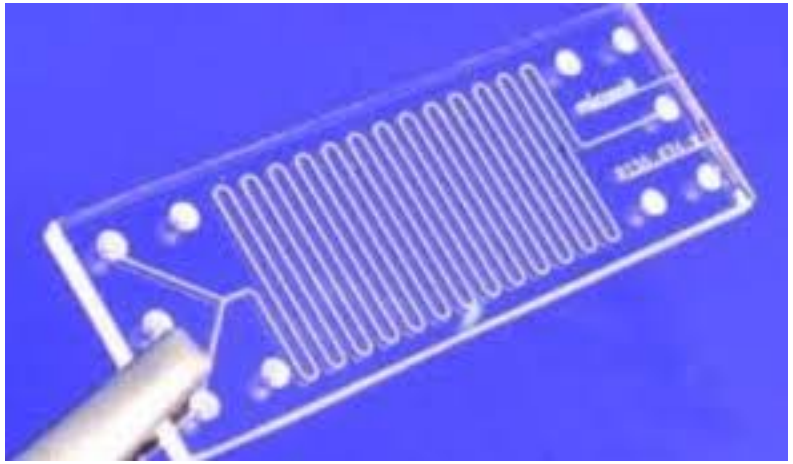
- "Tears of wine"
- Surface tension gradients
- Fingering phenomena

Scientific applications

- Aircraft de-icing
- Spreading on a solid substrate
- Film drainage in foams
- Surfactant replacement therapy for *Respiratory Distress Syndrome*
- Flow control in microfluidic devices



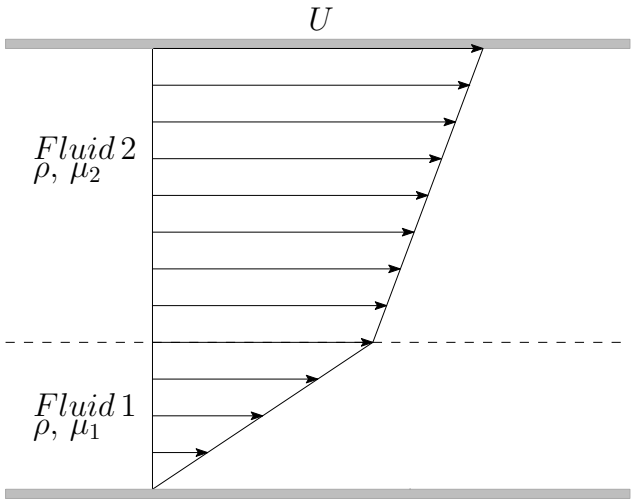




Two-layer shear flows: literature

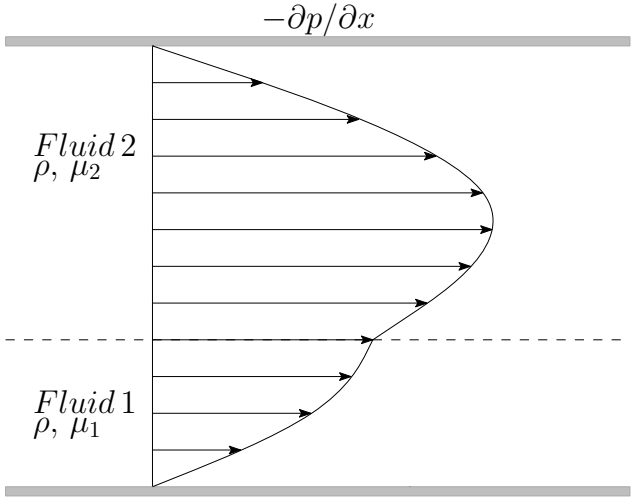
Linear:

- two-layer Couette-Poiseuille flow stable for $Re = 0$ [Yih, 1967]
- surfactant $\Rightarrow$ unstable even at $Re = 0$ [Frenkel & Halpern, 2002, 2003]
- destabilising effect of inertia [Blyth & Pozrikidis, 2004b]



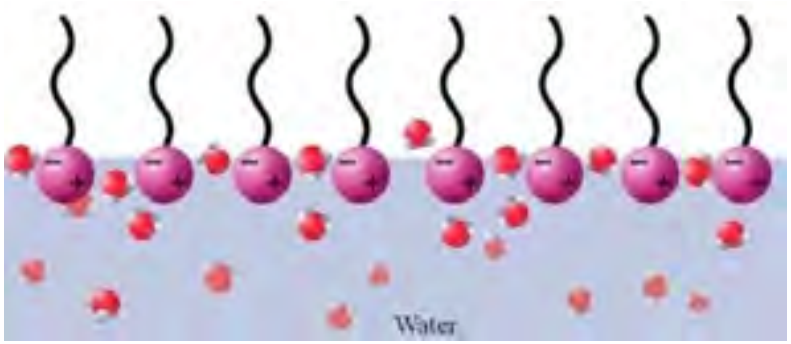
Nonlinear:

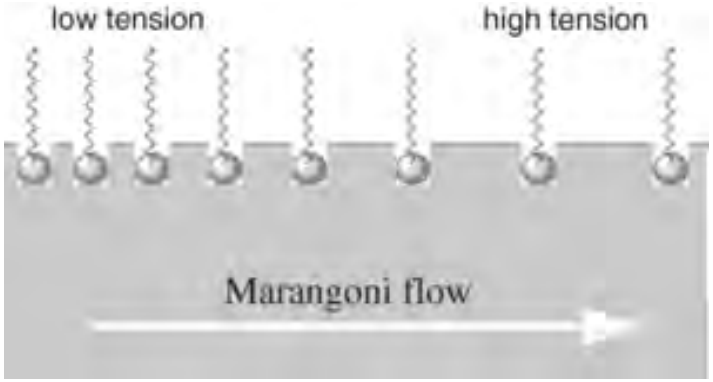
- short domains, $Re = 0$ [Blyth & Pozrikidis, 2004a]
- both layers thin, $Re = 0$ [Frenkel & Halpern, 2006]
- thin lower layer, local system [Bassom et al., 2010]
- nonlocal system with inertial effects [Kalogirou et al., 2012, 2016]
- effect of density stratification [Frenkel & Halpern, 2017]



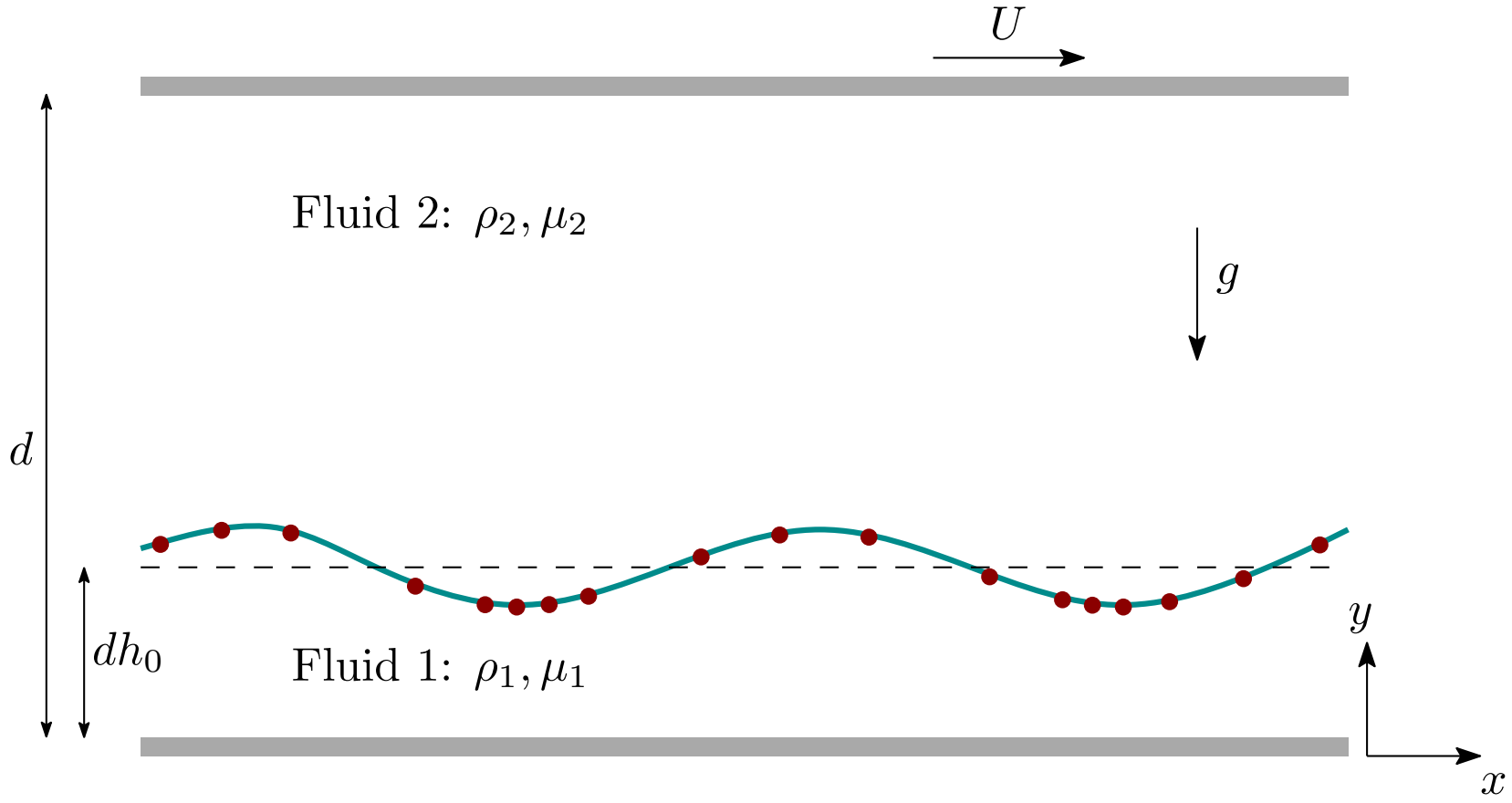
Objectives

- Instability of two-layer shear flows with one layer thin
- Presence of surfactant induces Marangoni effects
- Effects of gravity, inertia and Marangoni forces on dynamics
- Destabilisation mechanism
- Comparison with experiments
- Stability of one-dimensional travelling waves to spanwise perturbations





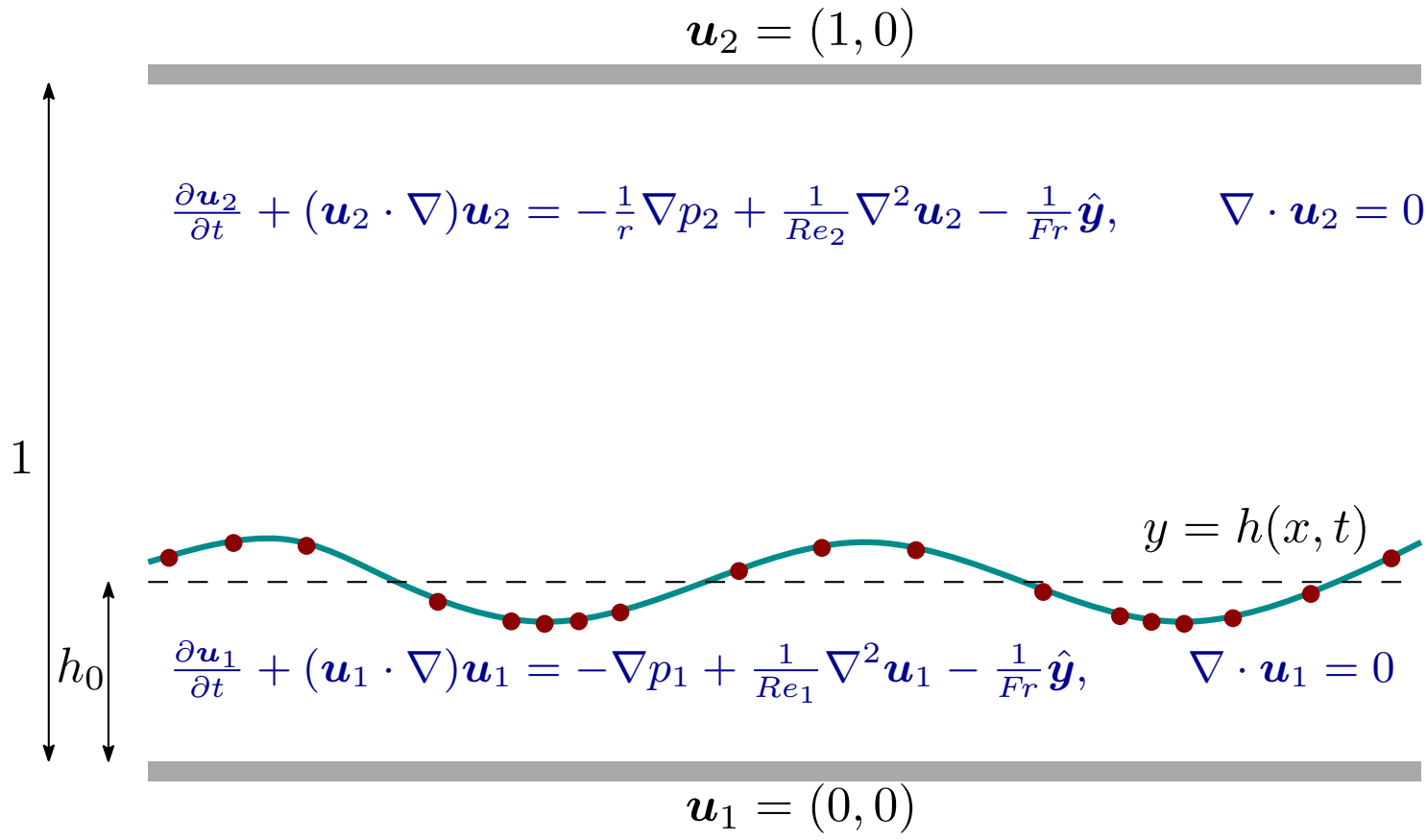
Problem configuration



Dimensionless parameters

$$r = \frac{\rho_2}{\rho_1}, \quad m = \frac{\mu_2}{\mu_1}, \quad Re_i = \frac{\rho_i U d}{\mu_i}, \quad Fr = \frac{U}{\sqrt{gd}}, \quad Ca = \frac{\mu_1 U}{\gamma_c}, \quad Ma = \frac{\beta}{Ca}, \quad Pe = \frac{U d}{\mathcal{D}_s}$$

Governing equations and boundary conditions



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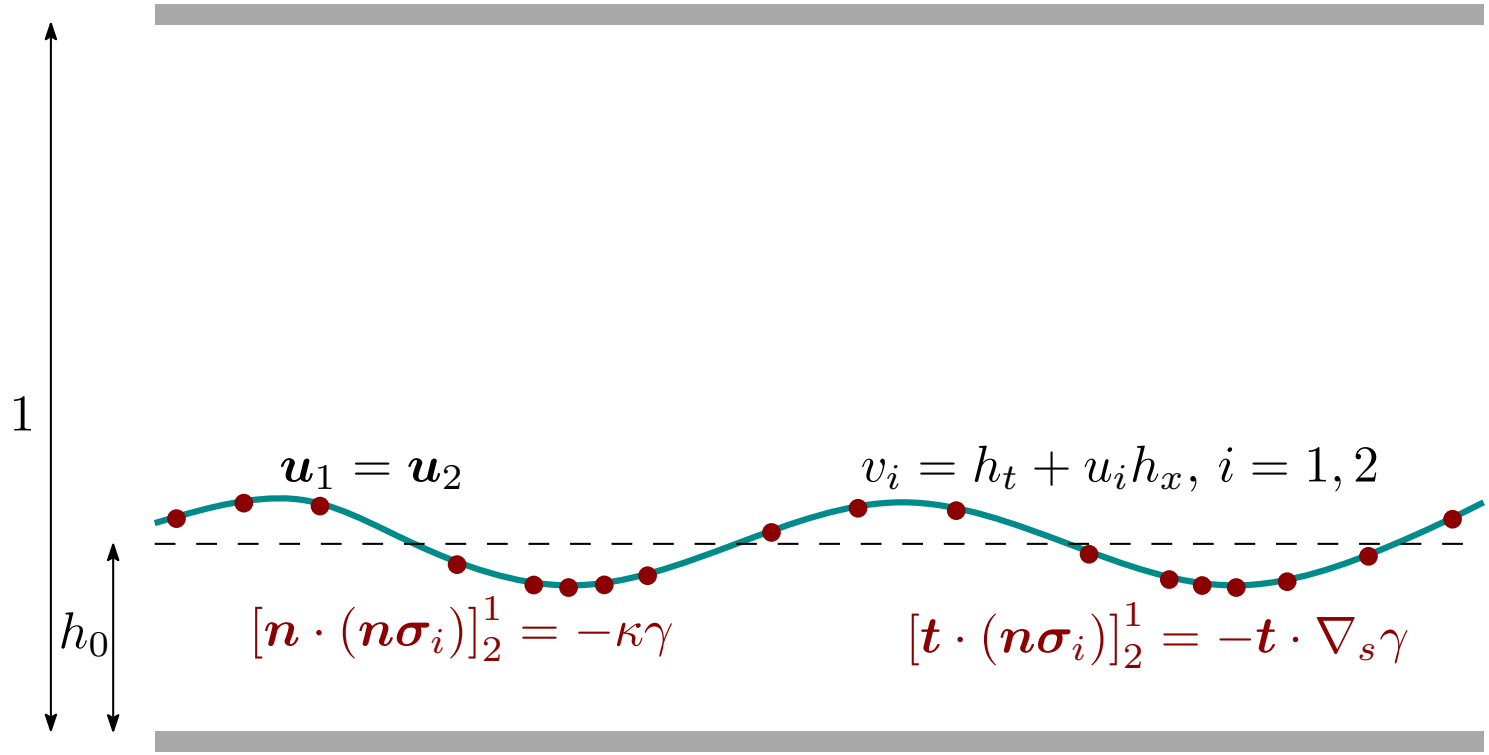
Internal conditions



Dimensionless parameters

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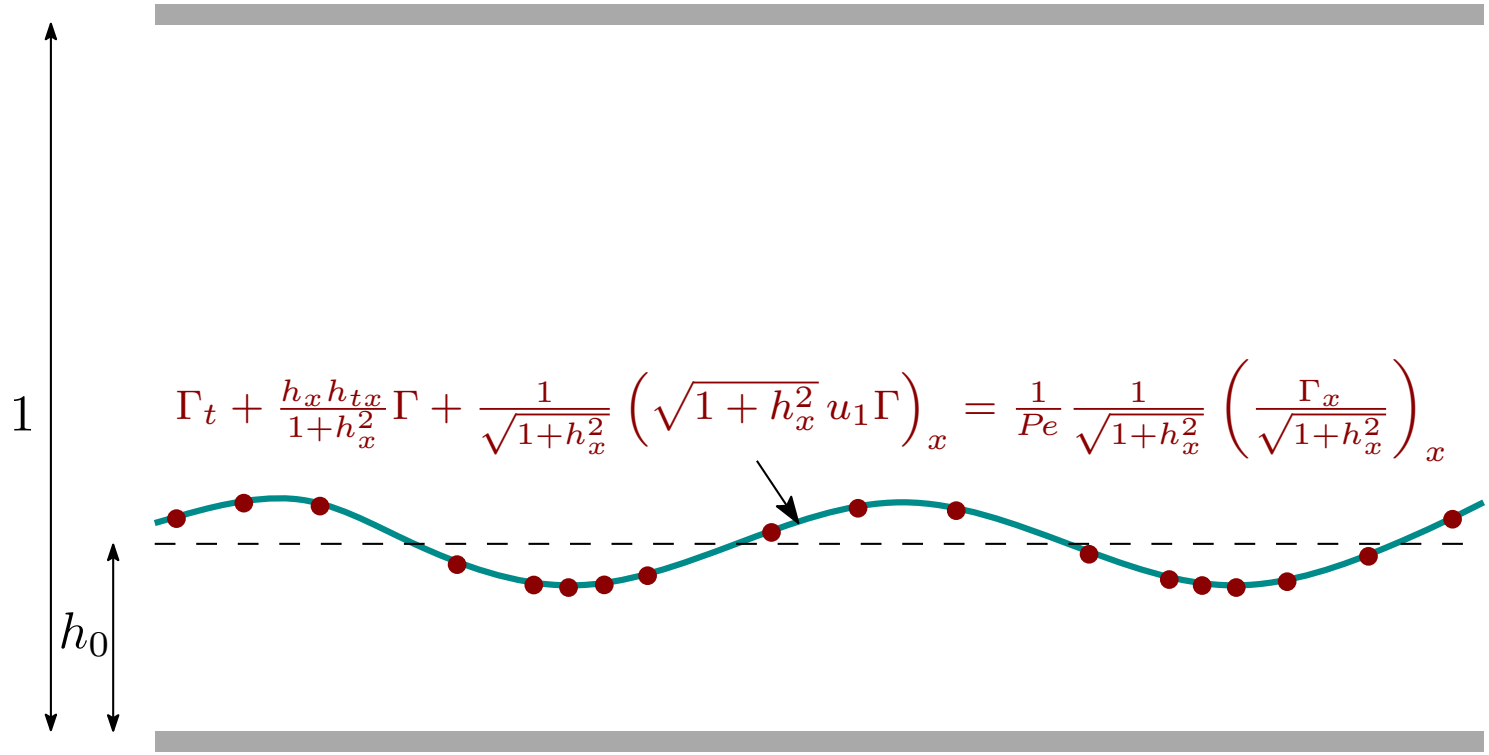
Surfactant transport equation



Dimensionless parameters

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Basic flow

Steady unidirectional flow

$$\bar{u}_1 = -\frac{1}{2}A_1 y^2 + B_1 y, \qquad \bar{v}_1 = 0, \quad \bar{p}_1 = P_0 - \frac{1}{Fr^2}(y - h_0) - Kx$$

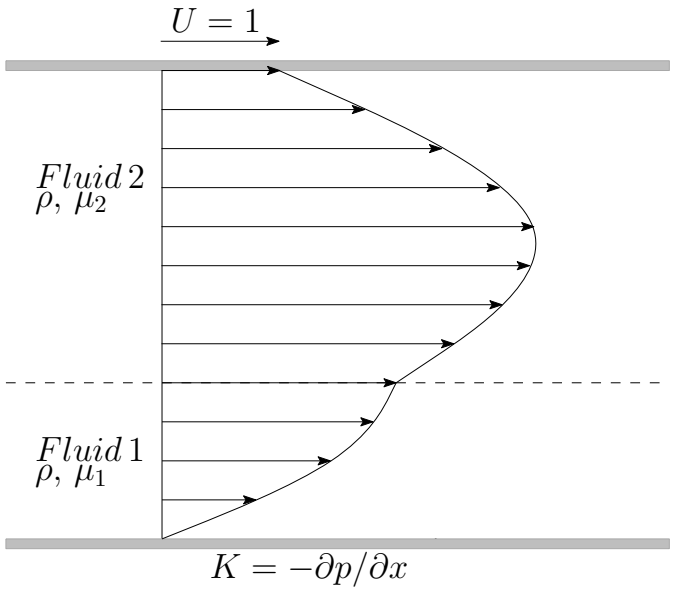
$$\bar{u}_2 = -\frac{1}{2}A_2 y^2 + B_2 y + C_2, \quad \bar{v}_2 = 0, \quad \bar{p}_2 = P_0 - \frac{r}{Fr^2}(y - h_0) - Kx$$

with constant P_0 and coefficients

$$A_1 = Re_1 K, \quad A_2 = \frac{1}{m} Re_1 K = \frac{A_1}{m},$$

$$B_1 = \frac{m + \frac{1}{2}A_1\big(1 + h_0^2(m-1)\big)}{1 + h_0(m-1)}, \quad B_2 = \frac{B_1}{m},$$

$$C_2 = \frac{m-1}{m}\left(\frac{mh_0 + \frac{1}{2}A_1 h_0(1-h_0)}{1 + h_0(m-1)}\right)$$



Thin-film limit: $h_0 = \epsilon \ll 1$

- Interfacial-shape and surfactant-concentration perturbations

$$h(x,t) = \epsilon + \epsilon^2 H(x,t),$$
$$\Gamma(x,t) = \delta(\epsilon)\tilde{\Gamma}(x,t).$$

$\rightarrow$ retain *nonlinearity* and *Marangoni effects* at leading order

- Lubrication approximations in film

$$u_1 = \overline{u}_1(y) + \epsilon^3 \tilde{u}_1 + \cdots, \qquad v_1 = \epsilon^4 \tilde{v}_1 + \cdots,$$
$$p_1 = \overline{p}_1(x,y) + \epsilon \tilde{p}_1 + \cdots.$$

- Linearized Navier-Stokes in upper fluid

$$u_2 = \overline{u}_2(y) + \epsilon^2 \tilde{u}_2 + \cdots, \qquad v_2 = \epsilon^2 \tilde{v}_2 + \cdots,$$
$$p_2 = \overline{p}_2(x,y) + \epsilon^2 \tilde{p}_2 + \cdots.$$

Solution in thin fluid 1

- Navier-Stokes equations

$$0 = -\frac{\partial \tilde{p}_1}{\partial x} + \frac{1}{Re_1}\frac{\partial^2 \tilde{u}_1}{\partial \xi^2}$$

$$0 = -\frac{\partial \tilde{p}_1}{\partial \xi}$$

$$\frac{\partial \tilde{u}_1}{\partial x} + \frac{\partial \tilde{v}_1}{\partial \xi} = 0$$

- Tangential stress balance

$$-\frac{\partial \tilde{u}_1}{\partial \xi}\bigg|_{\xi=\eta} + m\left(\frac{\partial \tilde{u}_2}{\partial y} + \frac{\partial \tilde{v}_2}{\partial x}\right)\bigg|_{y=\epsilon\eta} = Ma_0 \Gamma_x$$

where the Marangoni number has been rescaled as $Ma = \epsilon^2 \delta^{-1} Ma_0$

- Normal stress balance

$$\tilde{p}_1 = \frac{1}{Re_1 \overline{Ca}}\Big(Bo(\eta - 1) - \eta_{xx}\Big)$$

where the Capillary number has been rescaled as $Ca = \epsilon Ca_0$

Solution in thick fluid 2

Linearised Navier-Stokes equations

$$\bar{u}_2 \frac{\partial \tilde{u}_2}{\partial x} + \frac{d\bar{u}_2}{dy}\tilde{v}_2 = -\frac{1}{r}\frac{\partial \tilde{p}_2}{\partial x} + \frac{1}{R_e}\left(\frac{\partial^2 \tilde{u}_2}{\partial x^2} + \frac{\partial^2 \tilde{u}_2}{\partial y^2}\right)$$

$$\bar{u}_2 \frac{\partial \tilde{v}_2}{\partial x} = -\frac{1}{r}\frac{\partial \tilde{p}_2}{\partial y} + \frac{1}{R_e}\left(\frac{\partial^2 \tilde{v}_2}{\partial x^2} + \frac{\partial^2 \tilde{v}_2}{\partial y^2}\right)$$

$$\frac{\partial \tilde{u}_2}{\partial x} + \frac{\partial \tilde{v}_2}{\partial y} = 0$$

Move to Fourier space and define $\hat{\tilde{v}}_2 = -ik(m-1)\left(1+\frac{A_2}{2}\right)\hat{H}F(y)$

Orr-Sommerfeld type boundary-value problem

$$\left(F^{(iv)} - 2k^2F'' + k^4F\right) - ikR_e\left(\overline{u}_2\left(F'' - k^2F\right) - \bar{u}_2''F\right) = 0$$

$$F(0) = 0,\ F'(0) = 1, \quad F(1) = F'(1) = 0$$

Evolution equations

$$H_t + HH_x - BoH_{xx} + H_{xxxx} + i\Lambda \int_{-\infty}^{+\infty} \mathcal{N}(k; R_e)\hat{H}_k \, \mathrm{d}k - \Gamma_{xx} = 0$$

$$\Gamma_t + (H\Gamma)_x - \eta\Gamma_{xx} = 0$$

- Galilean translation and slow-time scale

$$x \to x - \overline{u}_1(\epsilon)t, \qquad t \to \epsilon^2 t$$

- Dimensionless parameters

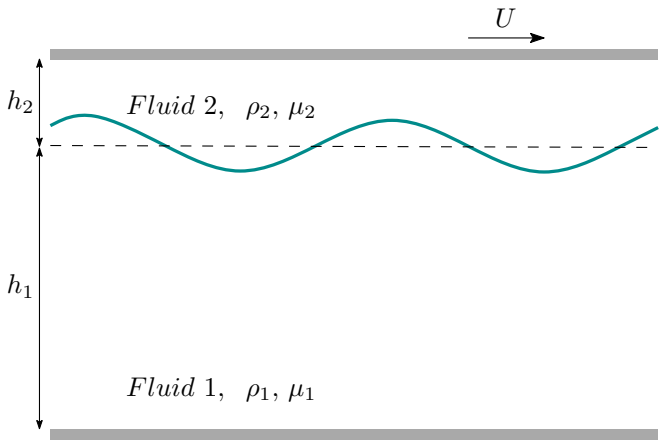
$$Bo = \frac{(1-r)d^2\rho_1 g}{\gamma_c}, \quad \Lambda = 3Ca_0\, m(1-m), \quad \eta = \frac{3Ca_0}{Pe_0}$$

- $\mathcal{N}(k; R_e) = -\frac{k}{2}F''(0) = \mathcal{N}_R + i\mathcal{N}_I$ effect of inertia in upper fluid layer

Validation of model evolution equation

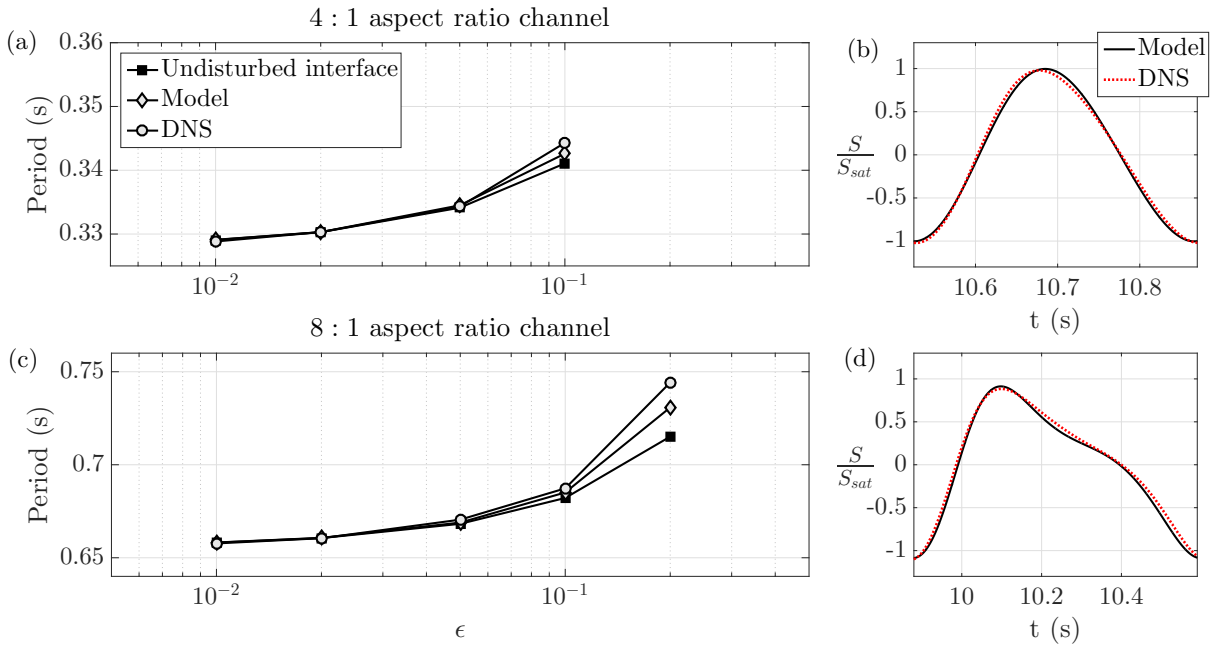
Film-thickness evolution equation

$$H_t - HH_x + H_{xxxx} - \frac{i\Lambda}{2\pi}\int_{-\infty}^{+\infty} \mathcal{N}(k;R_e)\hat{H}_k\,dk = 0$$

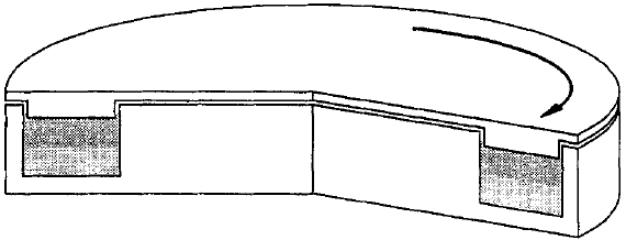


Destabilising mechanism: viscosity stratification and inertia

Track (normalised) deviation from flat interface at x^*

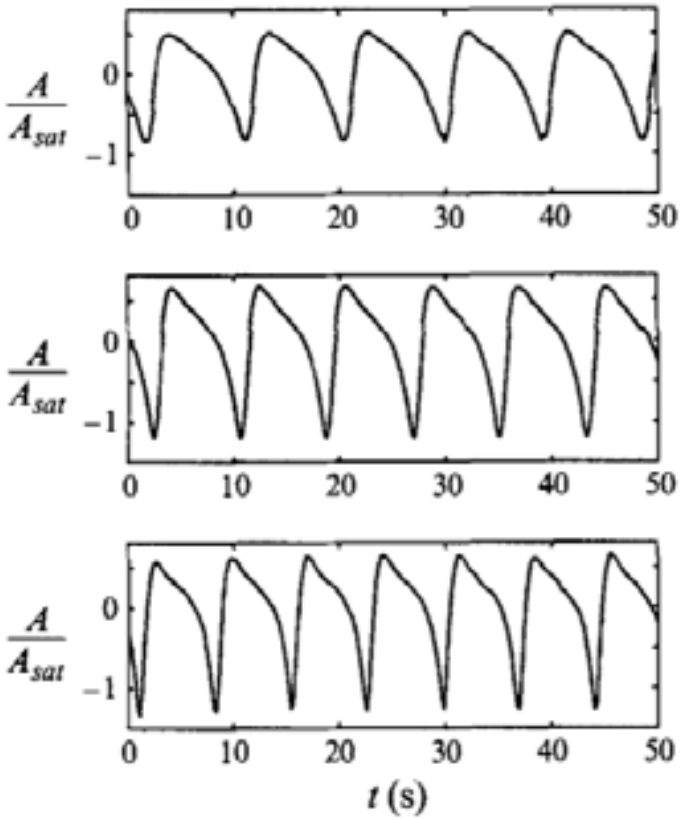


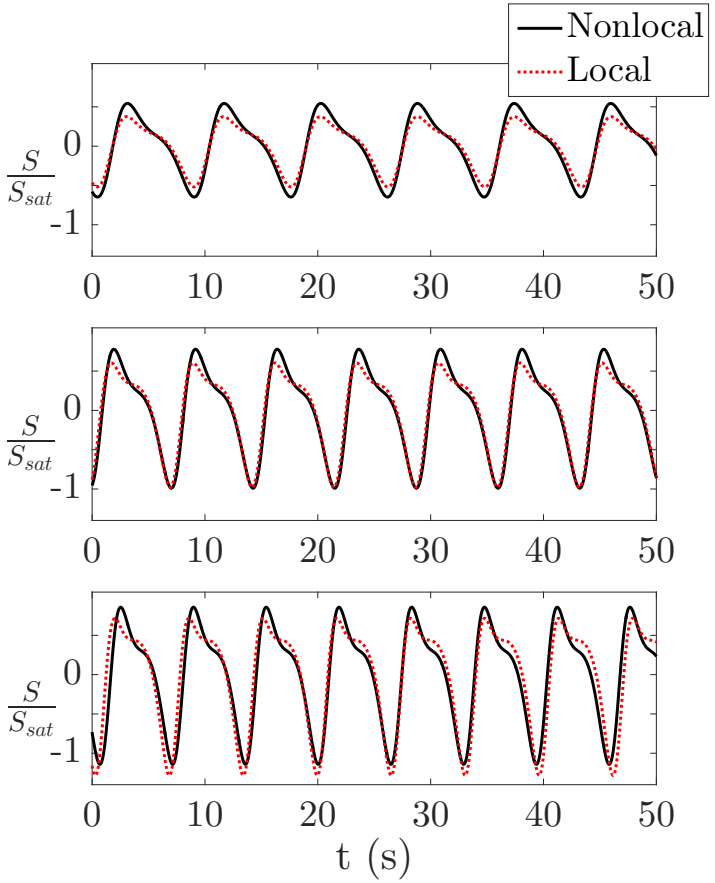
Comparison with experiment



[Barthelet, Charru & Fabre (1995)]

- Diameter $D = 400$ mm
- Width $W = 40$ mm
- Depth $d = 20$ mm
- $R_e = 389, 461, 516$
- $m = 2.76$
- $r = 0.741$
- $\epsilon = 0.2$





The effect of surfactant - Linear stability theory

Linearise around uniform states

$$H = 0 + \delta\,\hat{H}\,\mathrm{e}^{ikx+\sigma t},$$
$$\Gamma = \Gamma_0 + \delta\,\hat{\Gamma}\,\mathrm{e}^{ikx+\sigma t}, \qquad \delta \ll 1.$$

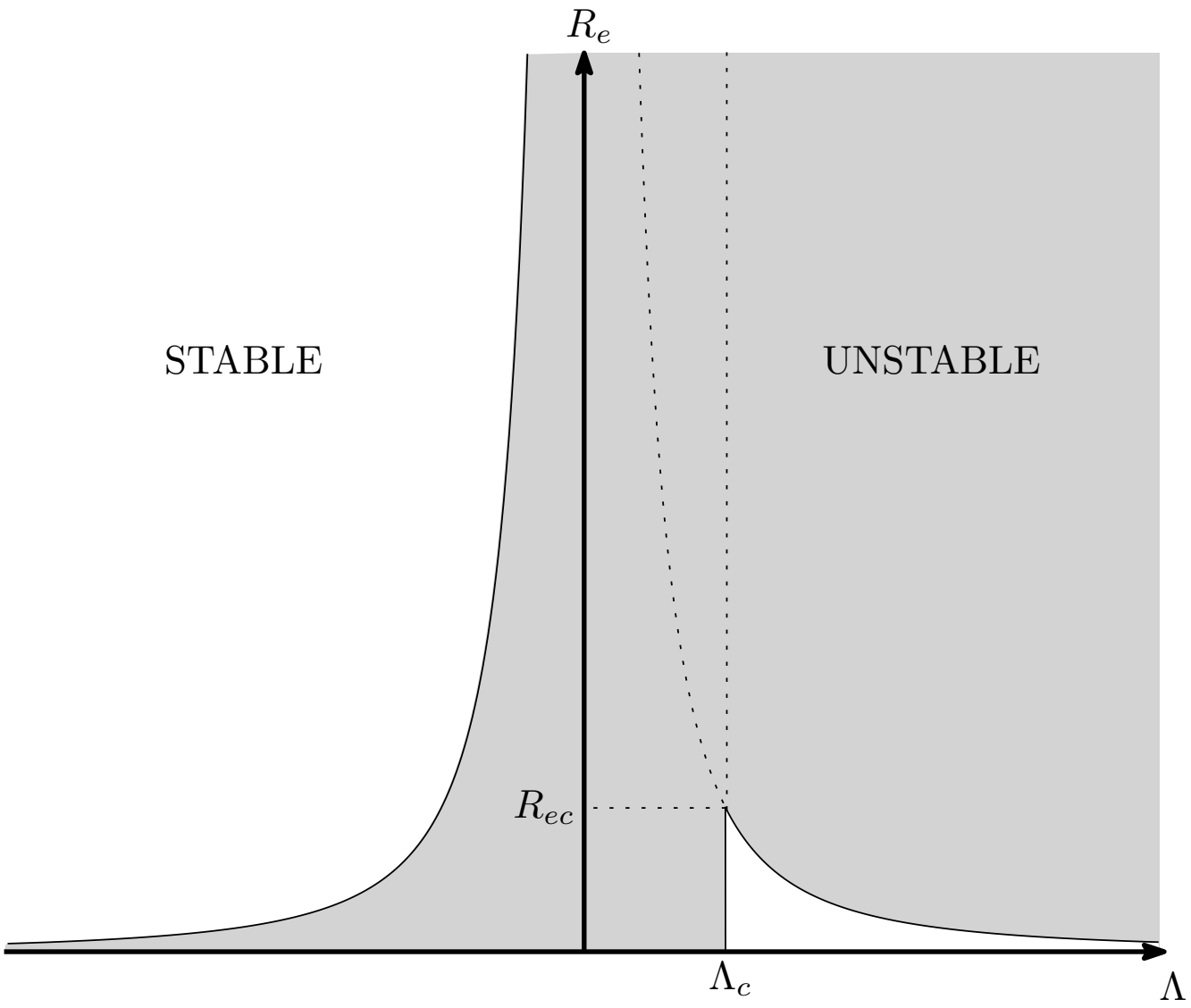
Dispersion relation

$$\sigma^2 + \left(Bok^2 + k^4 + i\Lambda\mathcal{N}(k) + \eta k^2\right)\sigma + \left(\eta Bok^4 + \eta k^6 + i\eta\Lambda\mathcal{N}(k)k^2 - ik^3\Gamma_0\right) = 0$$

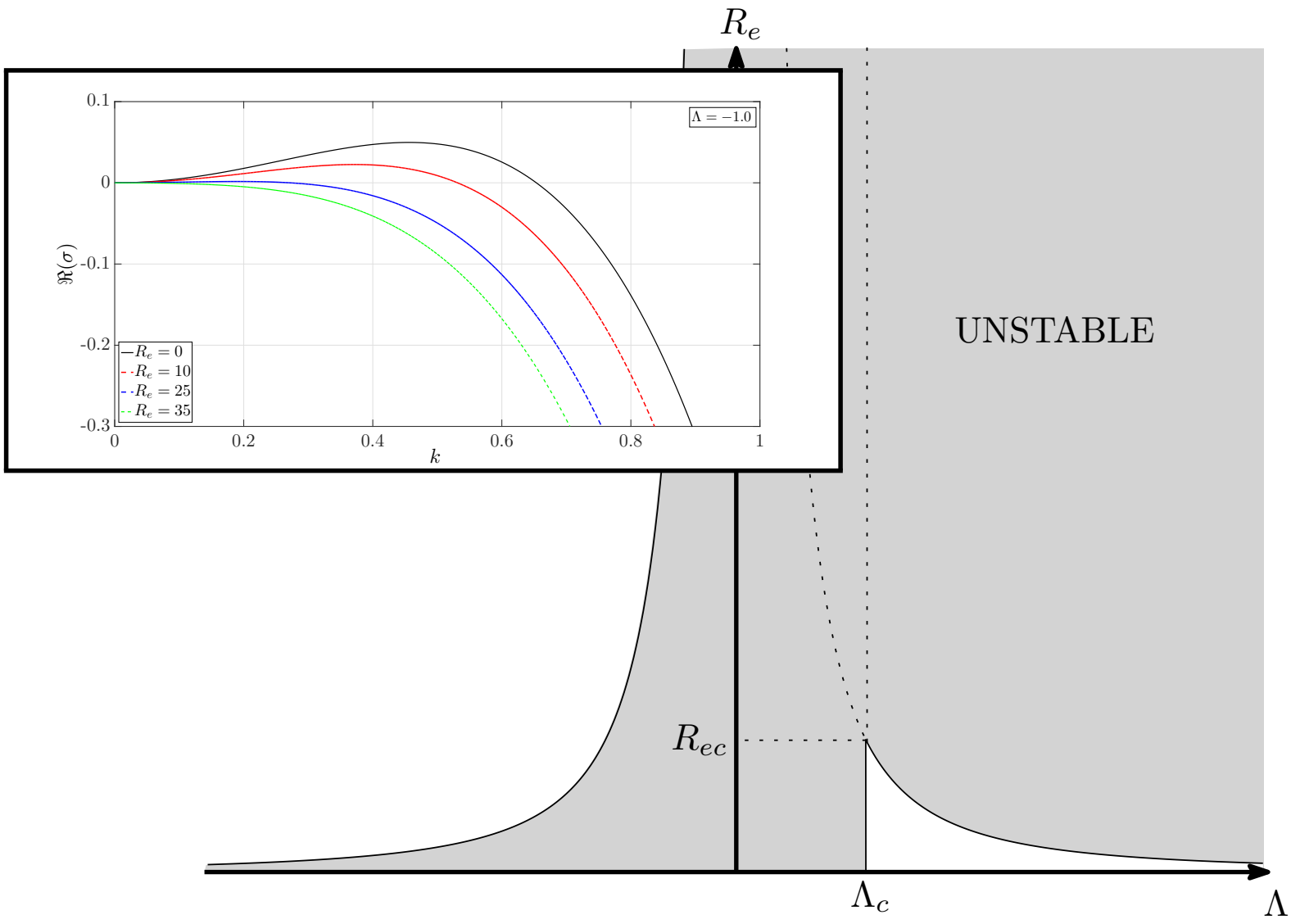
Amplification rates for $k \ll 1$

$$\Re(\sigma_1) = \left(\frac{\Gamma_0}{2\Lambda} - \eta\right)k^2, \qquad \Re(\sigma_2) = \left(\frac{\Lambda R_e}{60} - Bo - \frac{\Gamma_0}{2\Lambda}\right)k^2.$$

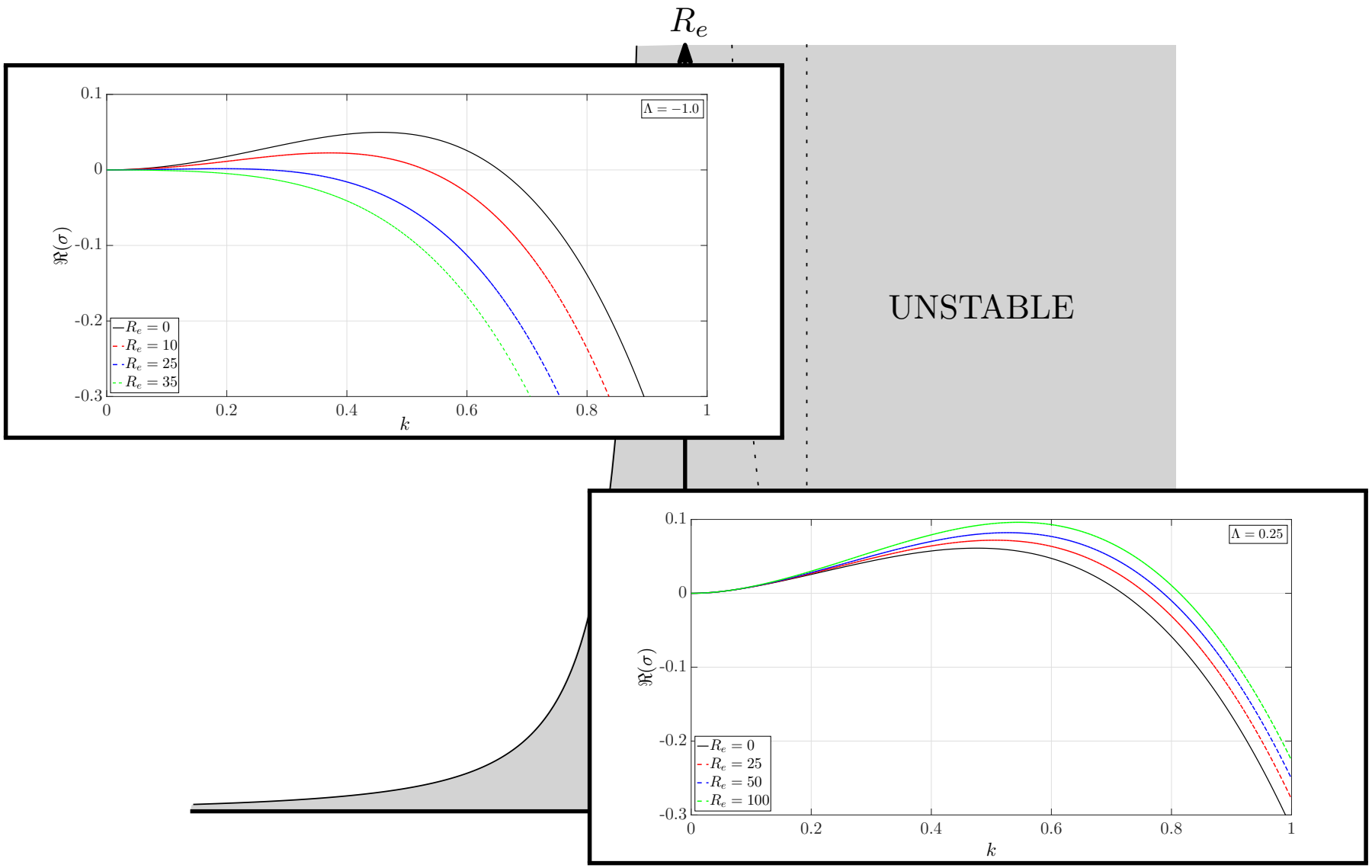
Linear stability diagram: $Bo = 0$



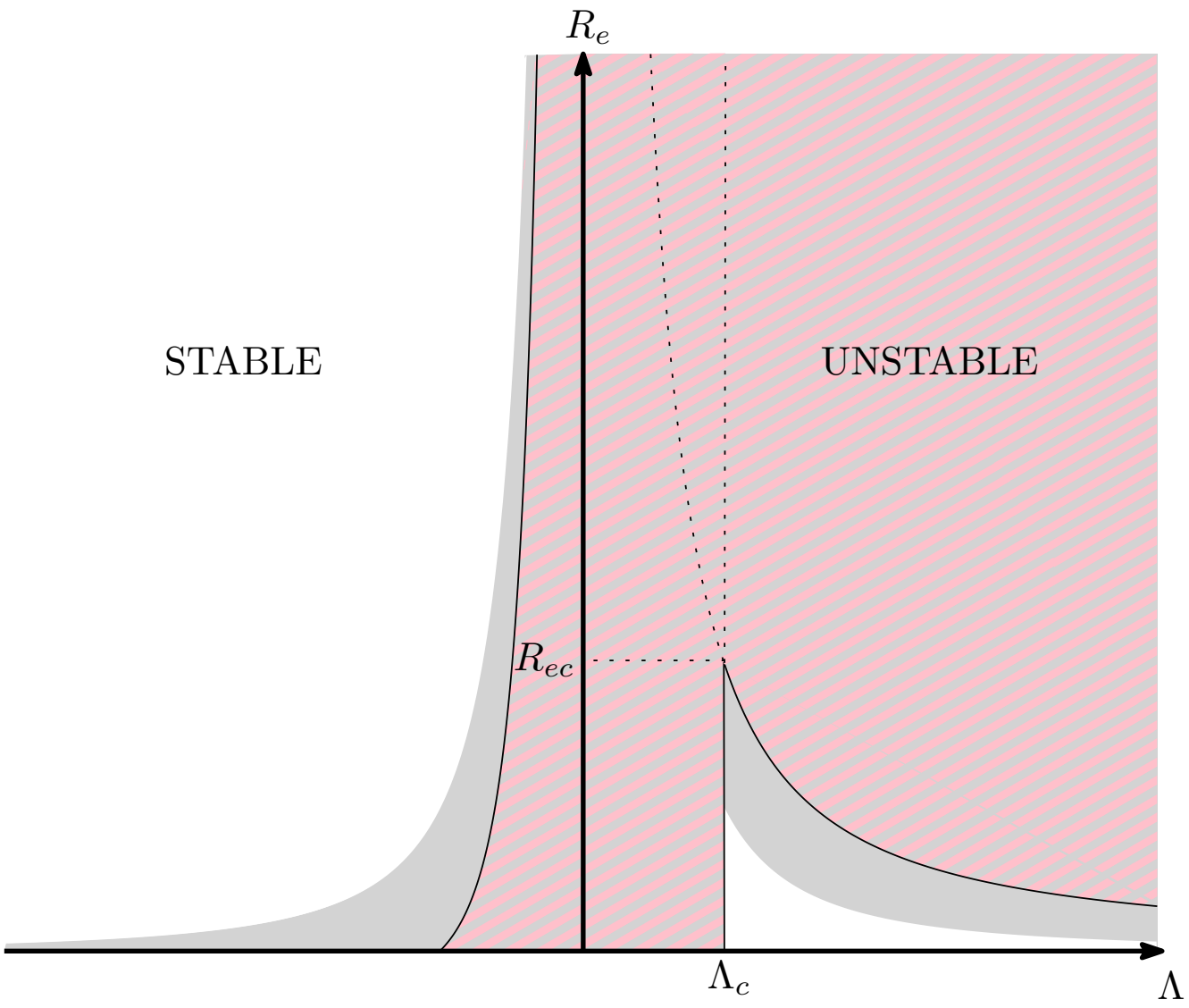
Linear stability diagram: $Bo = 0$



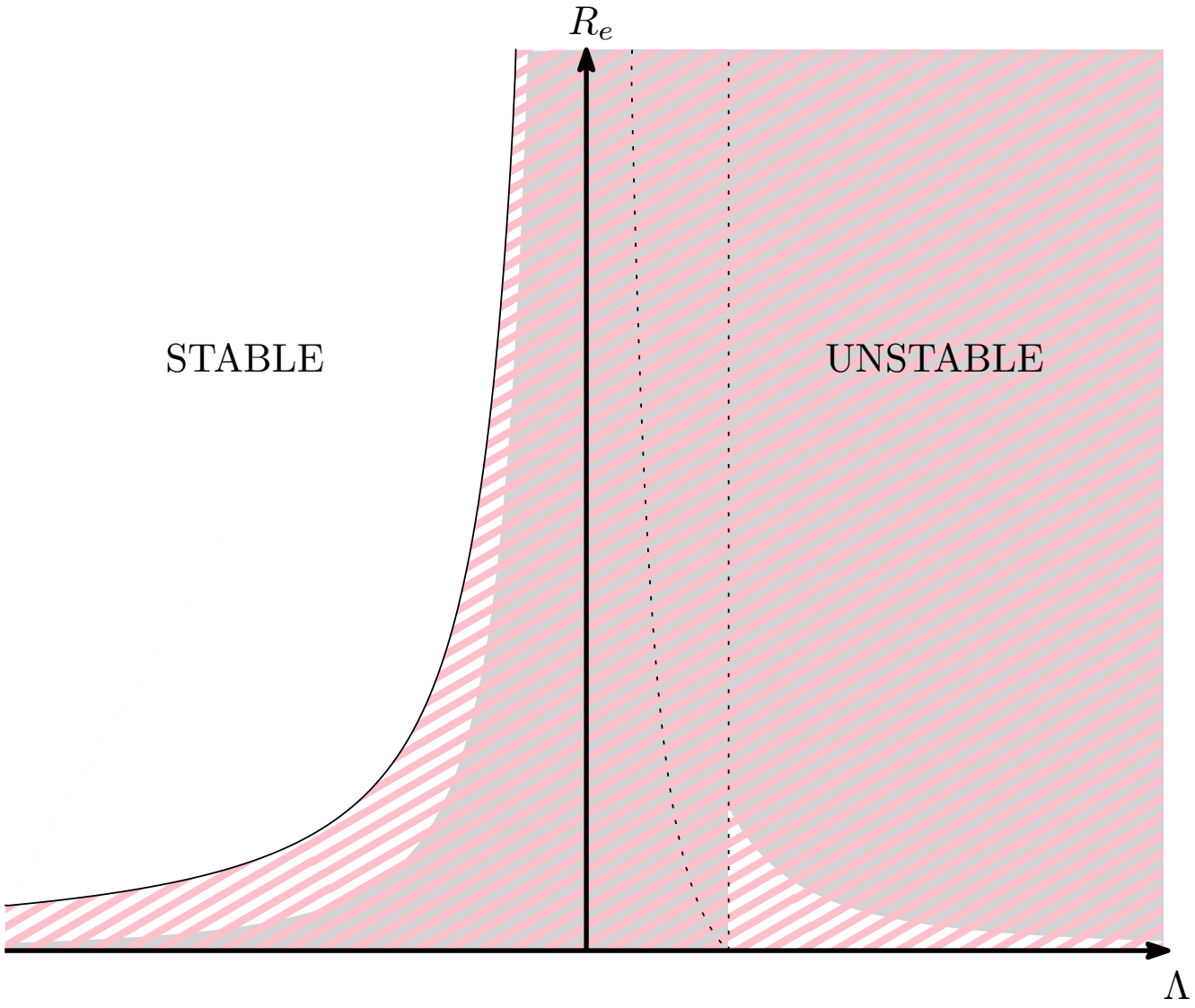
Linear stability diagram: $Bo = 0$



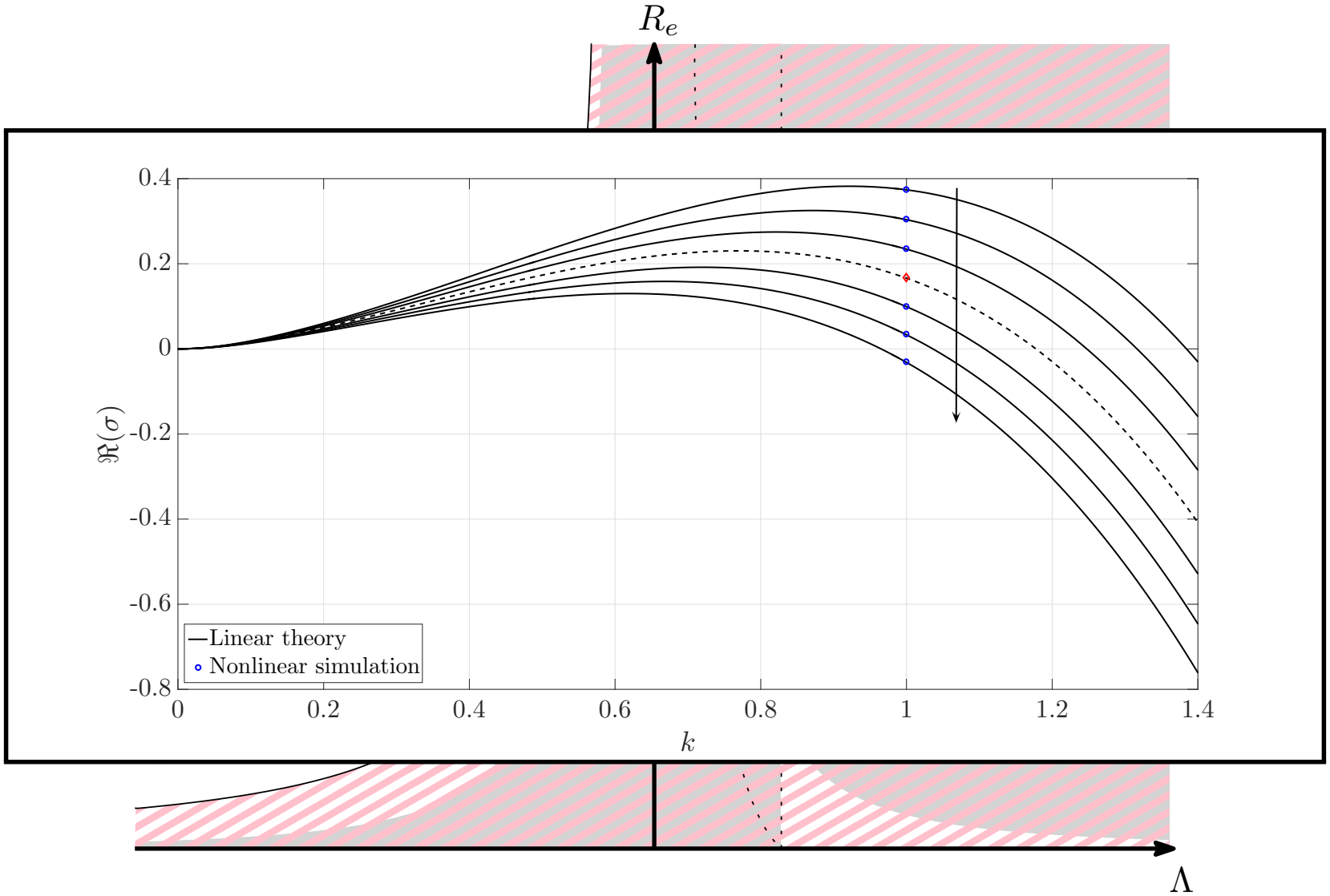
Linear stability diagram: $Bo > 0$



Linear stability diagram: $Bo < 0$

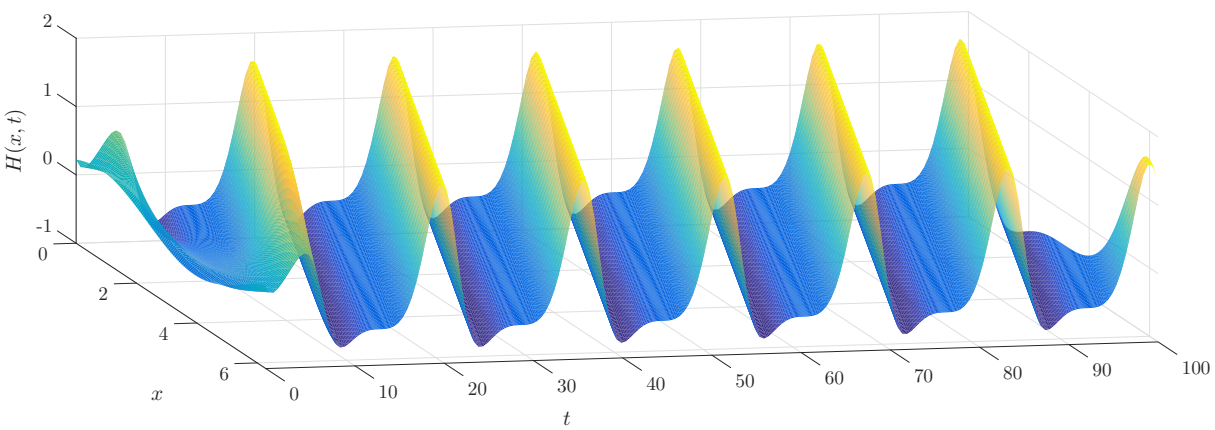


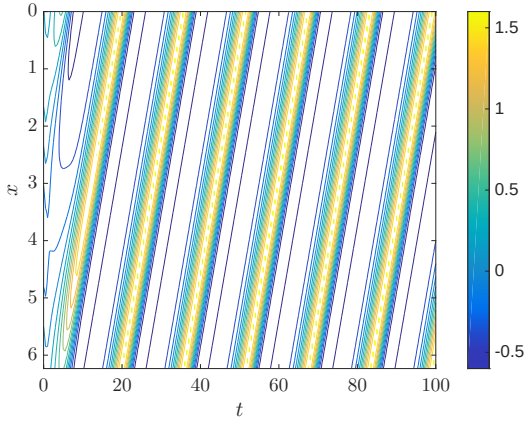
Linear stability diagram: $Bo < 0$

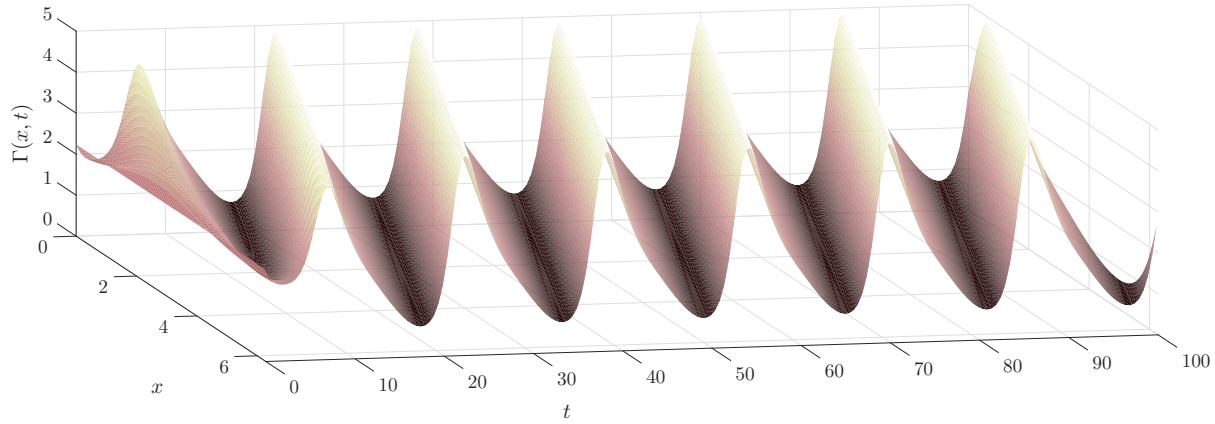


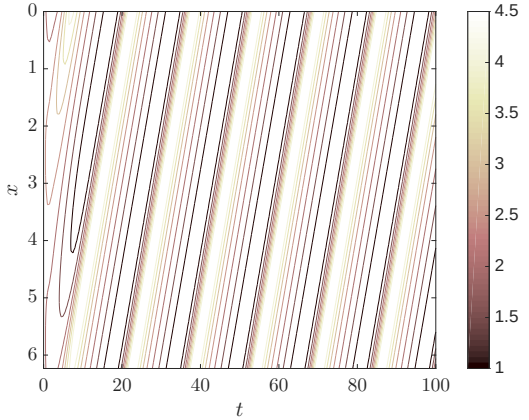
Travelling waves

Travelling waves



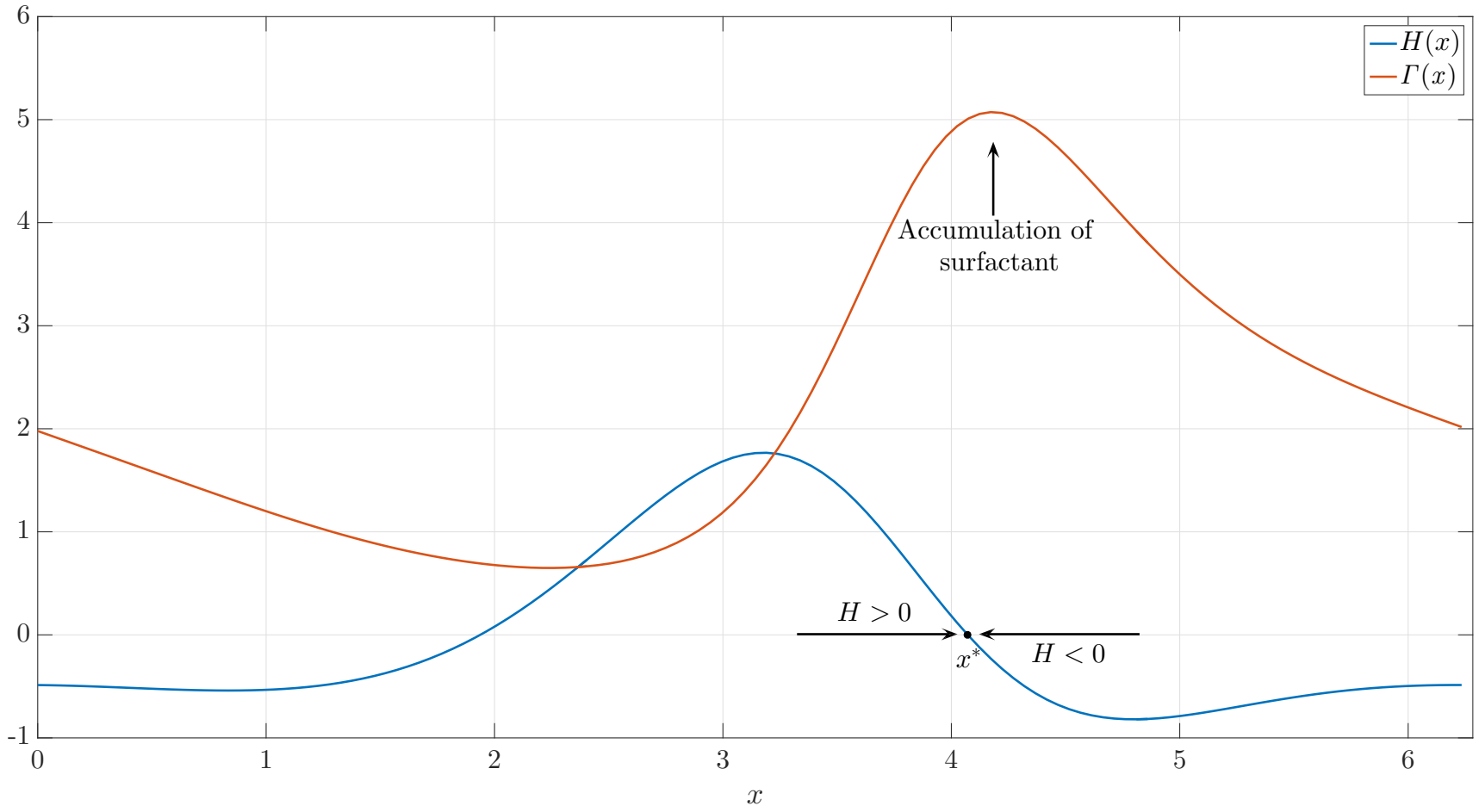




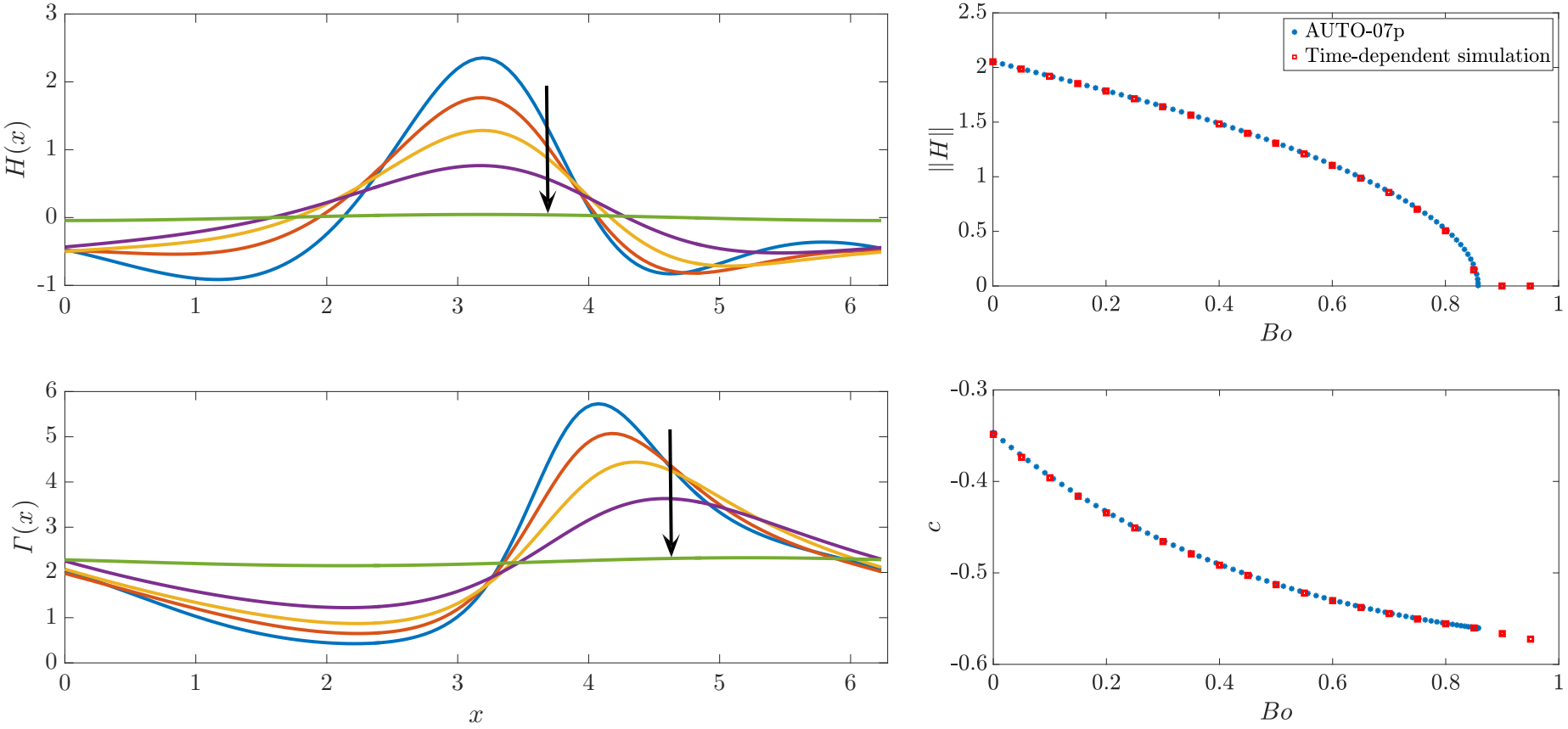


Destabilisation mechanism

$$u_1 = \overline{U}_1(y = \epsilon + \epsilon^2 H) + \epsilon^3 \tilde{u}_1 + \cdots = \overline{U}_1(\epsilon) + \underbrace{m\epsilon^2 H}_{\text{leading-order perturbation}} + \epsilon^3 \tilde{u}_1 + \cdots$$



Stabilisation due to gravity



Complex dynamical phenomena

$L = 7.0$

$L = 15.7$

$L = 37.5$

$L = 70.25$

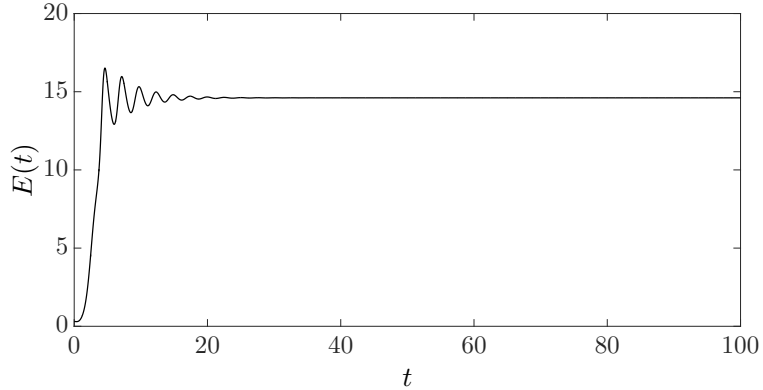
Complex dynamical phenomena

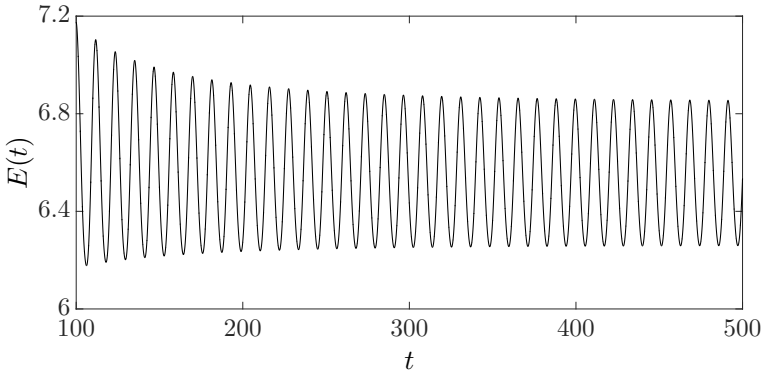
Signal energy

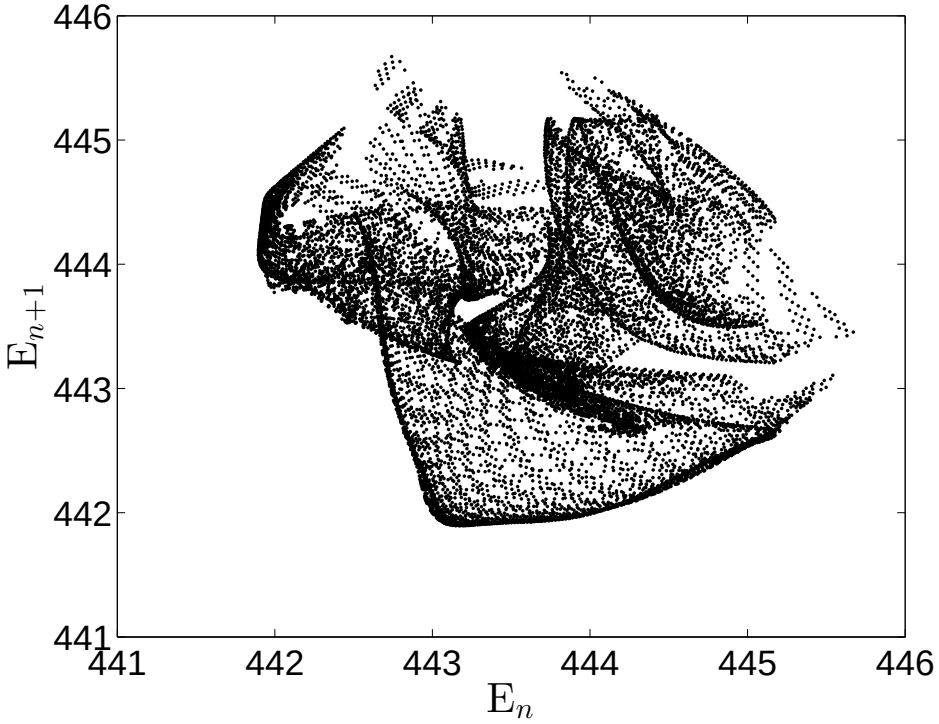
$$E(t) := \int_0^{2\pi} H^2(x,t)\,dx$$

Poincaré (or return) map

Consists of the points (E_n, E_{n+1}), $n = 1, 2, \ldots,$ where E_n is the nth minimum of the energy signal $E(t)$.







Three-dimensional flows

Can Squire's theorem be generalised in two-layer channel flow with surfactants?

Squire's Theorem

If a growing 3D disturbance can be found at a given Reynolds number, then a growing 2D disturbance exists at a lower Reynolds number.

- $R_e = 0$: Squire's theorem is valid
- $R_e \neq 0$: 3D flow more unstable

$$H_t + HH_x + \nabla^4 H + i\Lambda \iint_{-\infty}^{+\infty} \mathcal{N}(k_1, k_2; R_e) \hat{H}_{k_1,k_2} \, \mathrm{d}k_1 \, \mathrm{d}k_2 - \nabla^2 \Gamma = 0$$

$$\Gamma_t + (H\Gamma)_x - \eta \nabla^2 \Gamma = 0$$

Three-dimensional flows

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Stability of travelling waves

Perturb 1D travelling waves by

$$H(x,z,0) = \overline{H}(x - c\tau) + \mathcal{R}(z), \quad \Gamma(x,z,0) = \overline{\Gamma}(x - c\tau) + \mathcal{R}(z),$$

where

$$\mathcal{R}(z) = \sum_{m=1}^{N} (\alpha_m \sin mz + \beta_m \cos mz)$$

with random Fourier coefficients α_m, $\beta_m \in [-0.1, 0.1]$.

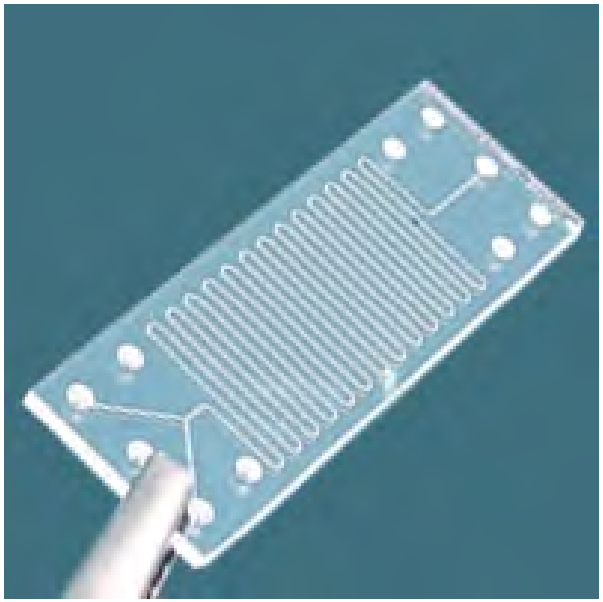
$L_1 = 7$, $L_2 = 7$

$L_1 = 7$, $L_2 = 18$

$L_1 = 7$, $L_2 = 34$

Conclusions

- Mathematical model for two-phase Couette-Poiseuille flows
- Takes into account gravitational, inertial and Marangoni effects
- Nonlocal term for upper layer dynamics
- Reproduces experimental results (in absence of surfactant)
- Exhibits travelling-wave solutions unstable to spanwise perturbations
- Destabilisation mechanism due to Marangoni forces



[Kalogirou et al. (2012), *IMA J. Appl. Math.*]

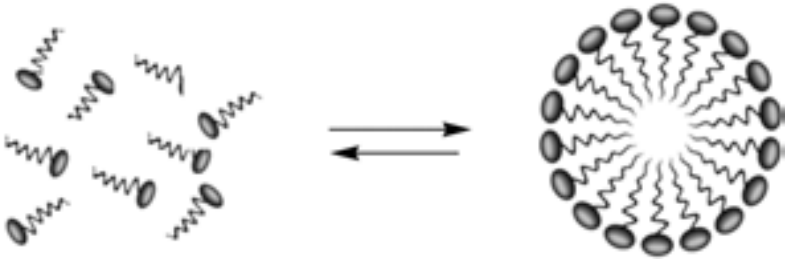
[Kalogirou & Papageorgiou (2016), *J. Fluid Mech.*]

[Kalogirou et al. (2016), *J. Fluid Mech.*]

[Kalogirou & Blyth, *in preparation*.]

Extensions

1. Effect of surfactant solubility: below and above *Critical Micelle Concentration*



2. Viscoelastic surfactant systems: rod-like micelles, affect viscosity, study flow stability

3. Flows in different geometries, *e.g.* pipes, inclined planes with topography

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