Imperial College London

AMCG

Rapid modelling in Reactor Physics using Reduced Order Modelling (ROM)

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The authors gratefully acknowledge financial support from the Defence Academy at HMS Sultan.

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1. Motivation

Aim

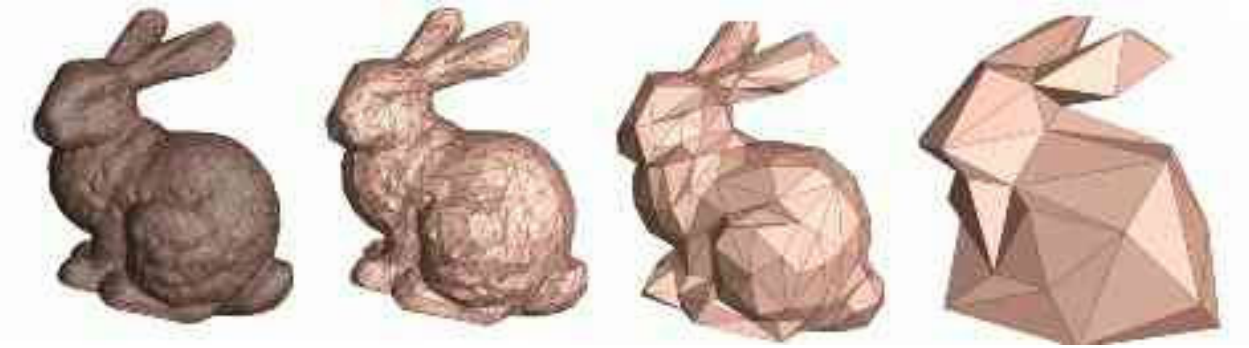
- (faster than) real time simulation of a nuclear reactor

Applicability

- reactor simulator for training purposes
- rapid response to particular events
- assimilated data
- optimisation and management
- uncertainty quantification

2. Reduced Order Modelling (ROM) in brief

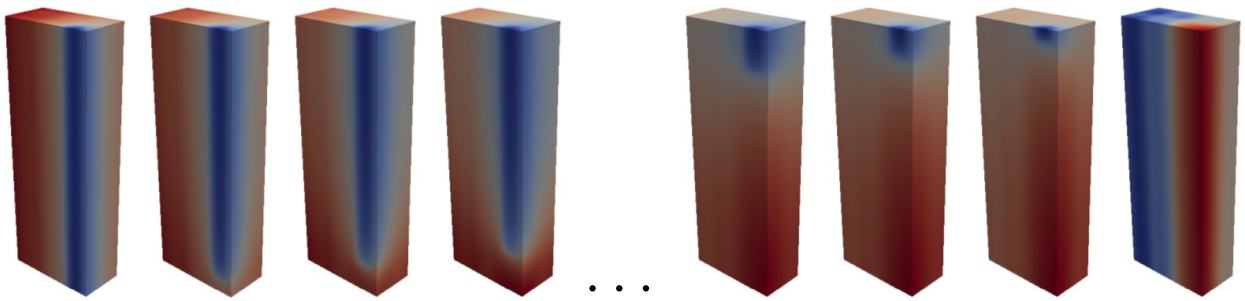
The aim of ROM is to achieve a compact (low-dimensional) description of a model's behaviour by reducing its complexity.



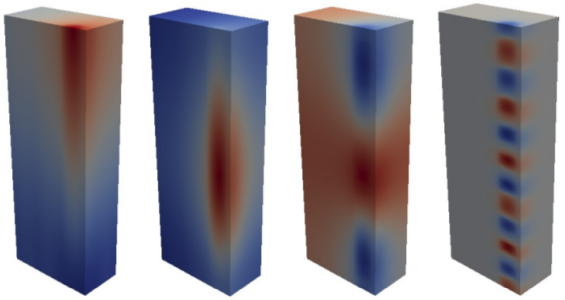
2. ROM in brief

Offline stage:

1. snapshots



2. basis functions



3. projection

Online stage:

4. construct the reduced system for the desired parameters

3. Governing Equations
Multigroup neutron diffusion equation

For group g, the criticality eigenvalue problem of the multigroup neutron diffusion equation is

$$\underbrace{-\nabla \cdot (D_g(x)\nabla\phi_g(x))}_{\text{leakage}} + \underbrace{\Sigma_g^a(x)\,\phi_g(x)}_{\text{absorption}} - \underbrace{\sum_{\substack{g'=1\\ g'\neq g}}^{G} \Sigma_{g'\to g}^s(x)\,\phi_{g'}(x)}_{\text{scattering}}$$

$$= \underbrace{\frac{\chi_g}{k^{\text{eff}}} \sum_{g'=1}^{G} \nu_{g'}\Sigma_{g'}^f(x)\,\phi_{g'}(x)}_{\text{fission}},$$

or $$A\phi = \frac{1}{k^{\text{eff}}}\,B\phi$$

Offline stage:

Full model solutions $\phi(\mu_j),\ j = 1, \ldots, S$

Snapshots matrix $M = [\phi(\mu_1) \quad \phi(\mu_2) \quad \cdots \quad \phi(\mu_S)]$

SVD $M = U \Sigma V^{\mathrm{T}}$

U contains the POD basis functions

Σ contains the singular values

POD basis $R =$ some columns of U

find the smallest value of $N^R \leqslant S$ such that

$$\frac{\sum_{i=1}^{N^R} \sigma_i^2}{\sum_{i=1}^{S} \sigma_i^2} > \texttt{tol}$$

4. ROM algorithm
ROM offline - projection

Projection using the basis functions R from the SVD of the snapshots

$$\phi = R\phi^{\mathrm{R}} \tag{1}$$

Full model problem

$$A\phi = \lambda B\phi \tag{2}$$

Reduced order model problem

$$\left(R^{\mathrm{T}}AR\right)\phi^{\mathrm{R}} = \lambda^{\mathrm{R}}\left(R^{\mathrm{T}}BR\right)\phi^{\mathrm{R}} \tag{3}$$

or

$$A^{\mathrm{R}}\phi^{\mathrm{R}} = \lambda^{\mathrm{R}}B^{\mathrm{R}}\phi^{\mathrm{R}} \tag{4}$$

4. ROM algorithm
ROM offline / online

Project the matrices associated with each set of parameters

$$A_i^{\mathrm{R}} = R^{\mathrm{T}} A(\mu_i) R \tag{5}$$

Online stage: For a control rod height of $\tilde{h} \in [h_i,\, h_{i+1}]$

$$A^{\mathrm{R}} = \omega A_i^{\mathrm{R}} + (1-\omega) A_{i+1}^{\mathrm{R}} \tag{6}$$

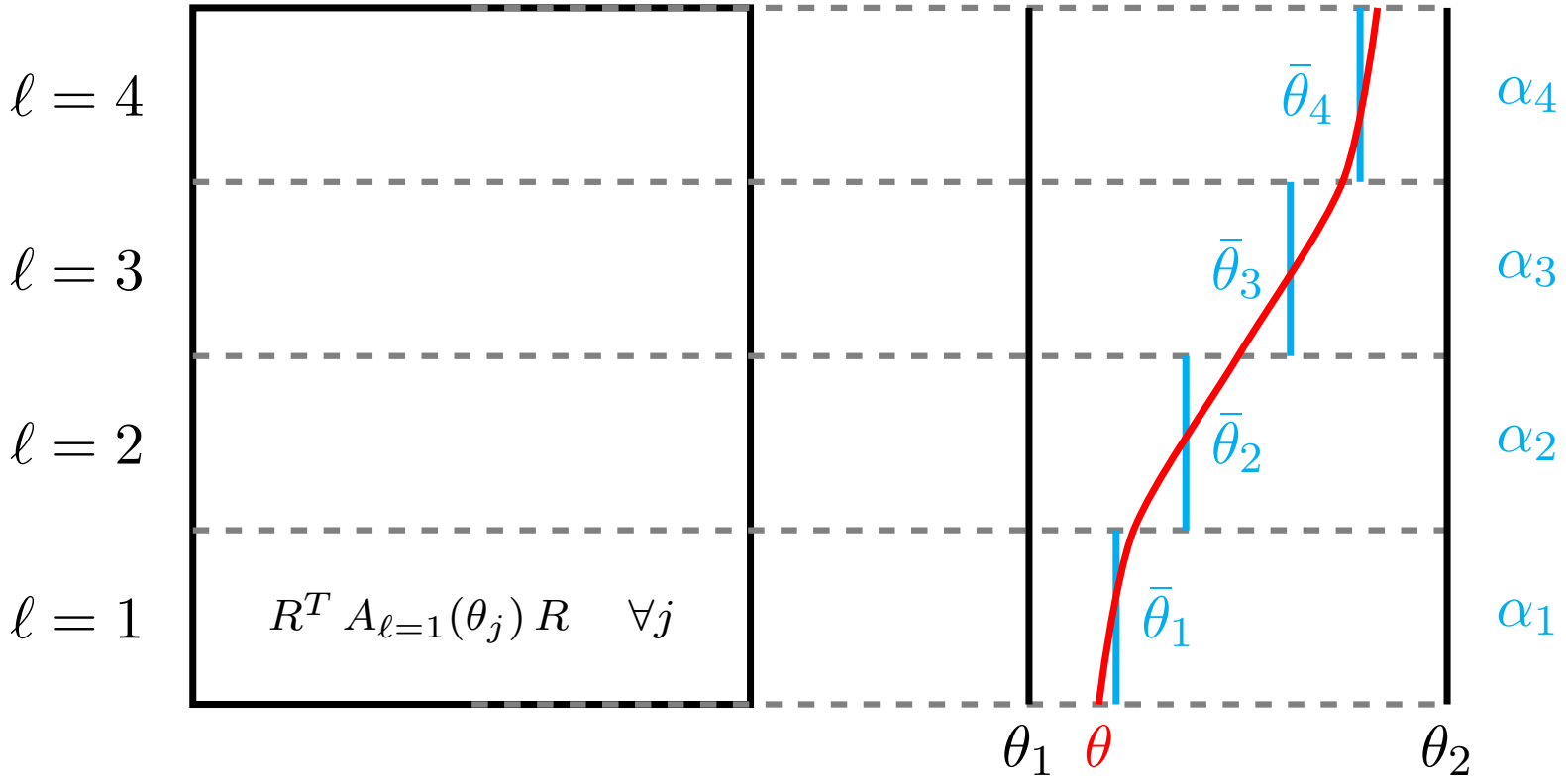
where, for linear interpolation, $\omega = \dfrac{h_{i+1} - \tilde{h}}{h_{i+1} - h_i}$.

B^{R} is assembled similarly and we solve

$$A^{\mathrm{R}} \phi^{\mathrm{R}} = \lambda^{\mathrm{R}} B^{\mathrm{R}} \phi^{\mathrm{R}} \ . \tag{7}$$

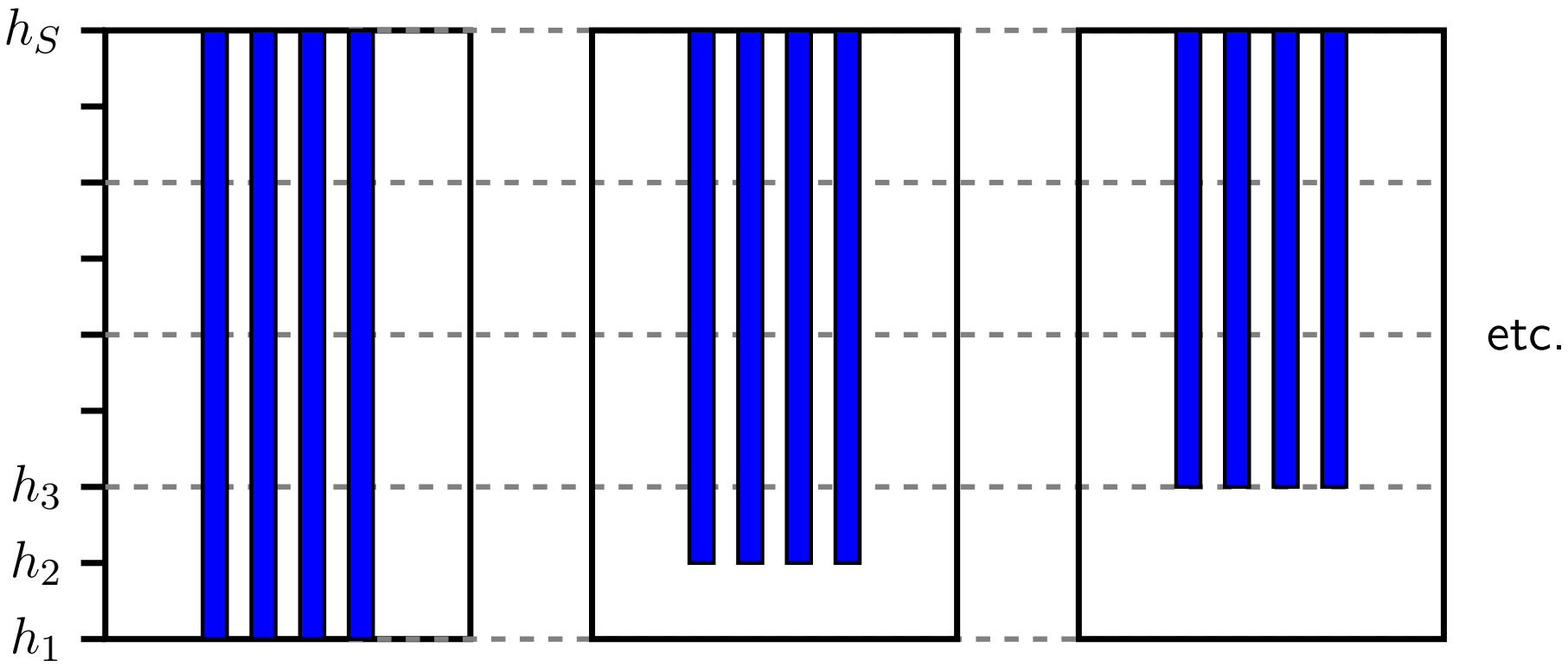
4. ROM algorithm

Imposing varying temperature profiles on the ROM



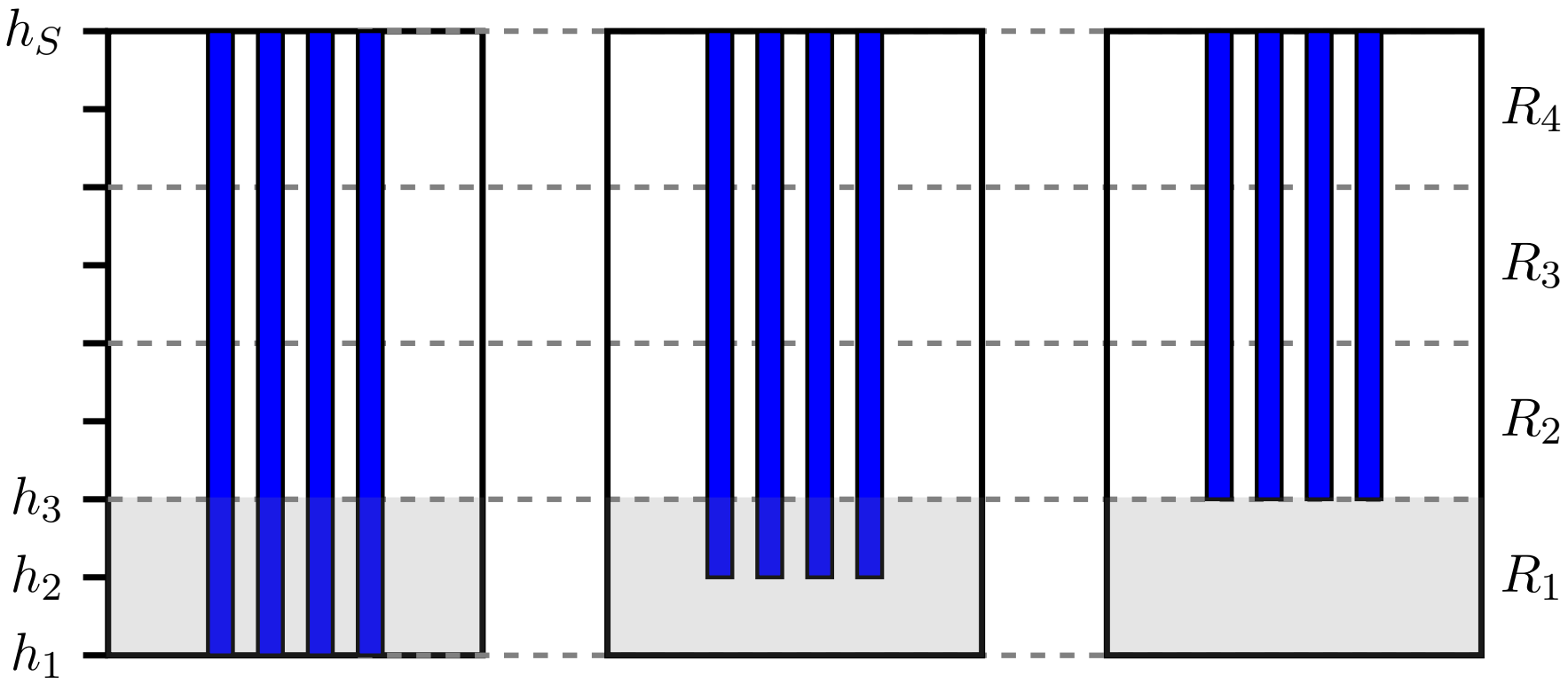
$$A^R = \alpha_1 \left(R^T\, A_1(\theta_1)\, R \right) + (1-\alpha_1) \left(R^T\, A_1(\theta_2)\, R \right) + \ \ldots$$

4. ROM algorithm
Global ROM algorithm



$$\text{SVD}[\phi(h_1)\ \phi(h_2)\ \cdots\ \phi(h_S)] = U\Sigma V^T$$

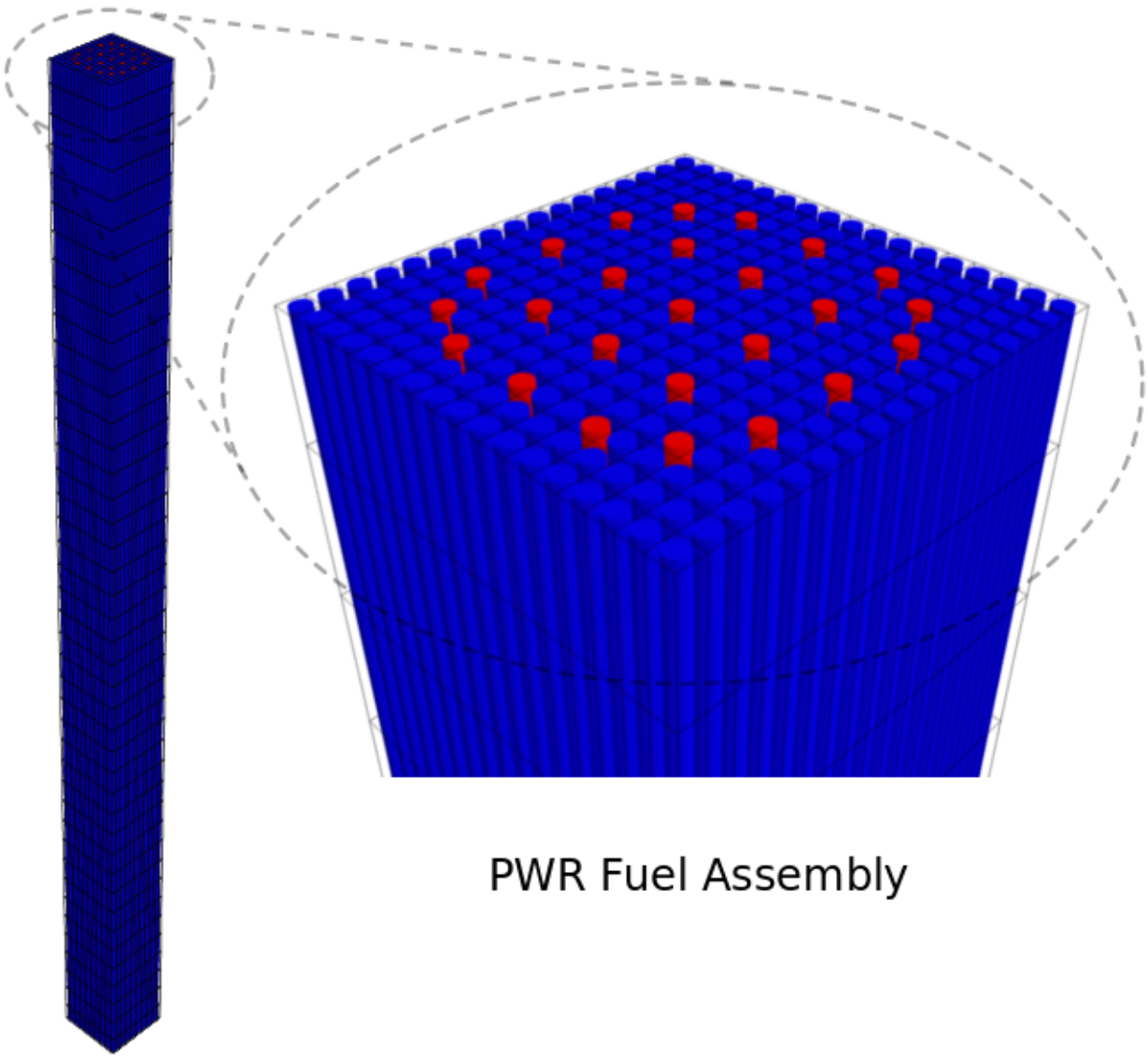
$$R = U(:,\, 1 : N^R)$$



$$\text{SVD}[\phi(h_1)\ \phi(h_2)\ \phi(h_3)] = U\Sigma V^T$$

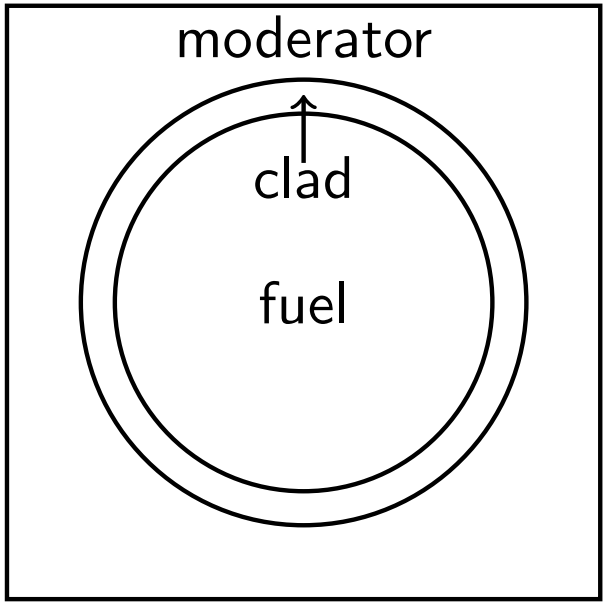
$$R_1 = U(:,\, 1 : N^R)$$

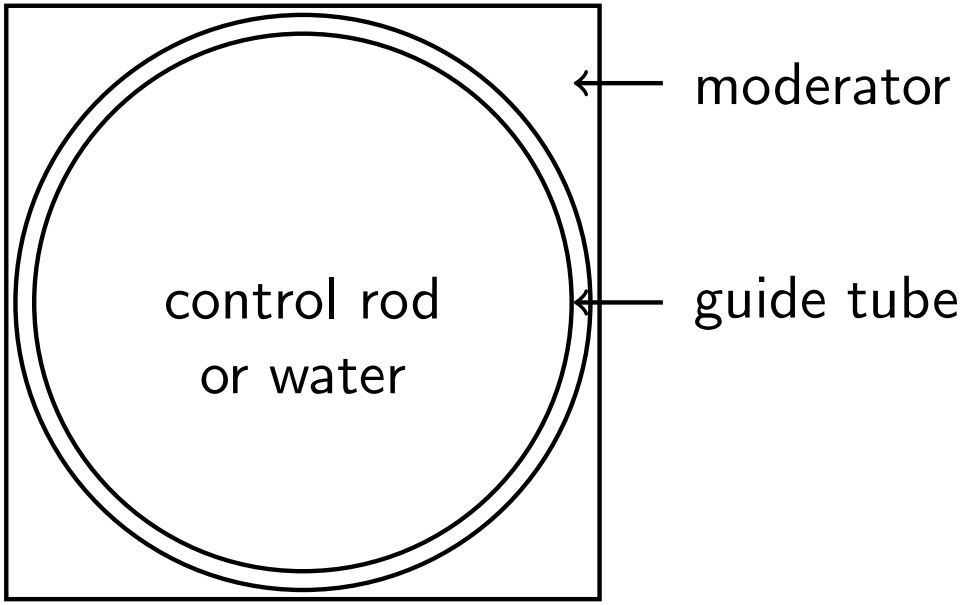
5. Results - Homogenised fuel assembly



5. Results - Homogenised fuel assembly

Fuel pins and control rod regions





5. Results - Homogenised fuel assembly

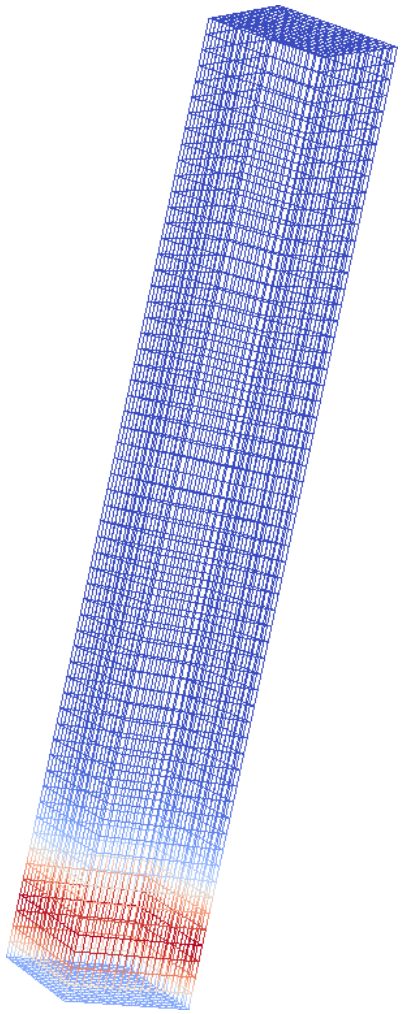
Full model

Finite element discretisation in space

Linear hexahedral elements

19,764 nodes

39,528 DoFs



5. Results - Homogenised fuel assembly

Problem: Criticality eigenvalue problem for multigroup neutron diffusion, $A\phi = \lambda B\phi$.

Offline stage:

- solve FETCH for 31 rod positions and 2 uniform temperature fields (62 snapshots):

$$\begin{aligned} h &\in \{0,\ 10,\ \cdots 290,\ 300\} \quad (\mathrm{cm}), \\ \theta &\in \{291,\ 1398\} \quad (^o\mathrm{C}). \end{aligned}$$

- calculate the basis functions
- form the reduced order model matrices

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Online stage:

- solve the reduced order model problem for a particular control rod height and temperature

ROM results:

- Control rod height increment $\Delta h = 1$cm.
- A number of temperature profiles are tested in the reduced order model:

$$\begin{aligned} \theta &= 291 \\ \theta &= 1398 \\ \theta &= 800 \\ \theta(z) &= 600 + 400 \sin\left(\frac{\pi z}{300}\right) \end{aligned}$$

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\theta &= 1398 \\
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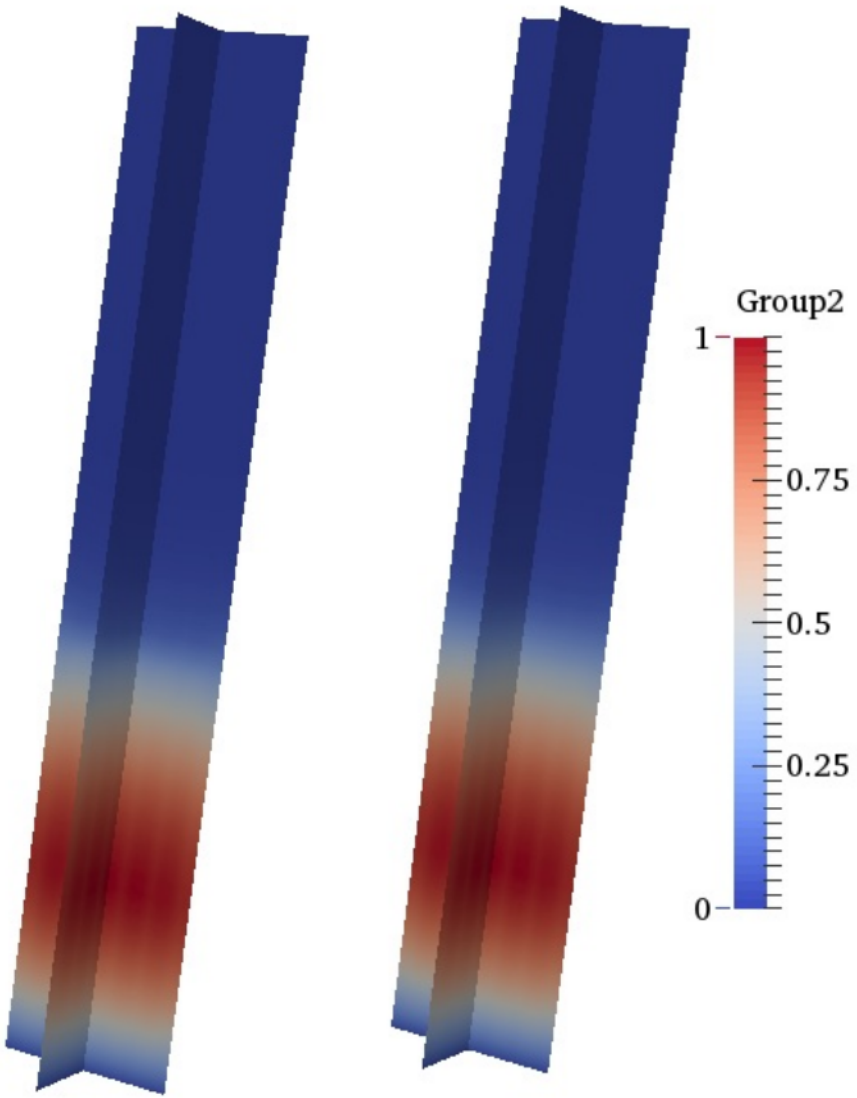
\end{aligned}
$$

5. Results - Homogenised fuel assembly

Group 2 fluxes. Left, full model, right, ROM (h=125 cm) - global meth



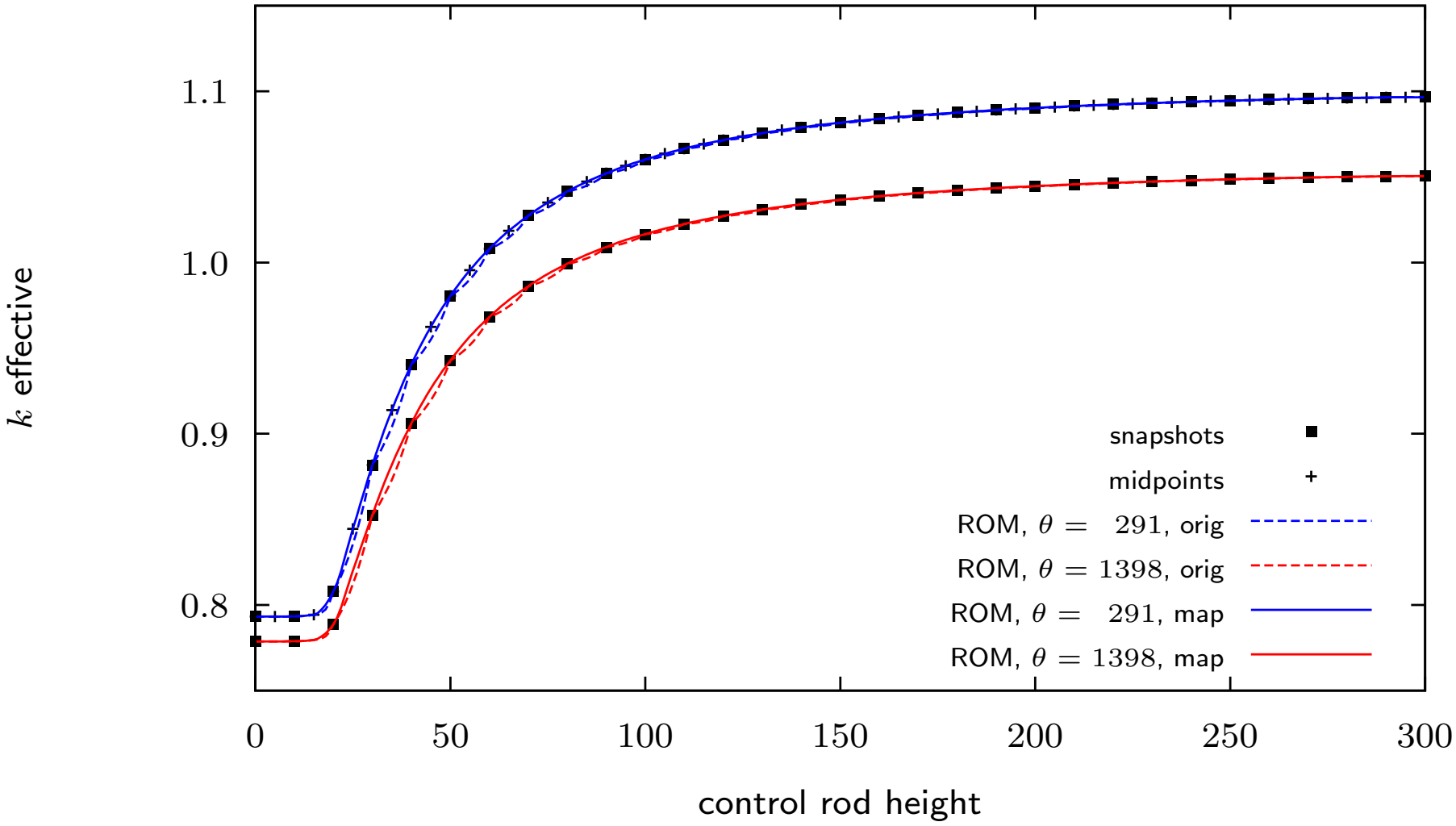
Results - Homogenised fuel assembly

Number of basis functions for each group (tol 0.999995)

		group 1	group 2
Global		17 of 62	25 of 62
Local	layer 1	5 of 8	6 of 8
	2	5	5
	3	4	4
	4	4	4
	5	4	4
	6	3	4
	7	3	4
	8	3	4
	9	3	4
	10	3	3

5. Results - Homogenised fuel assembly

Values of k effective of the snapshots and ROM - global method



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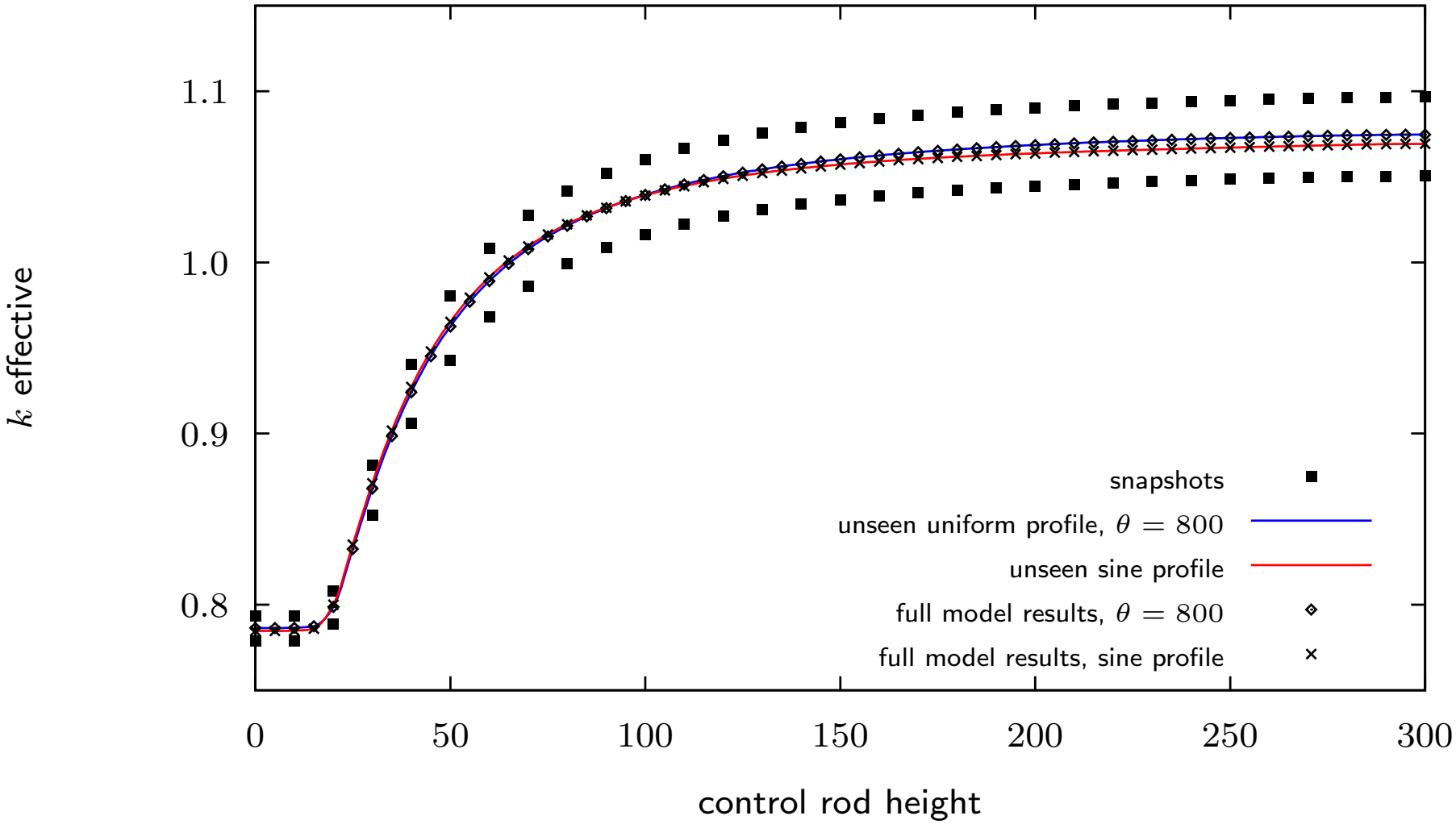

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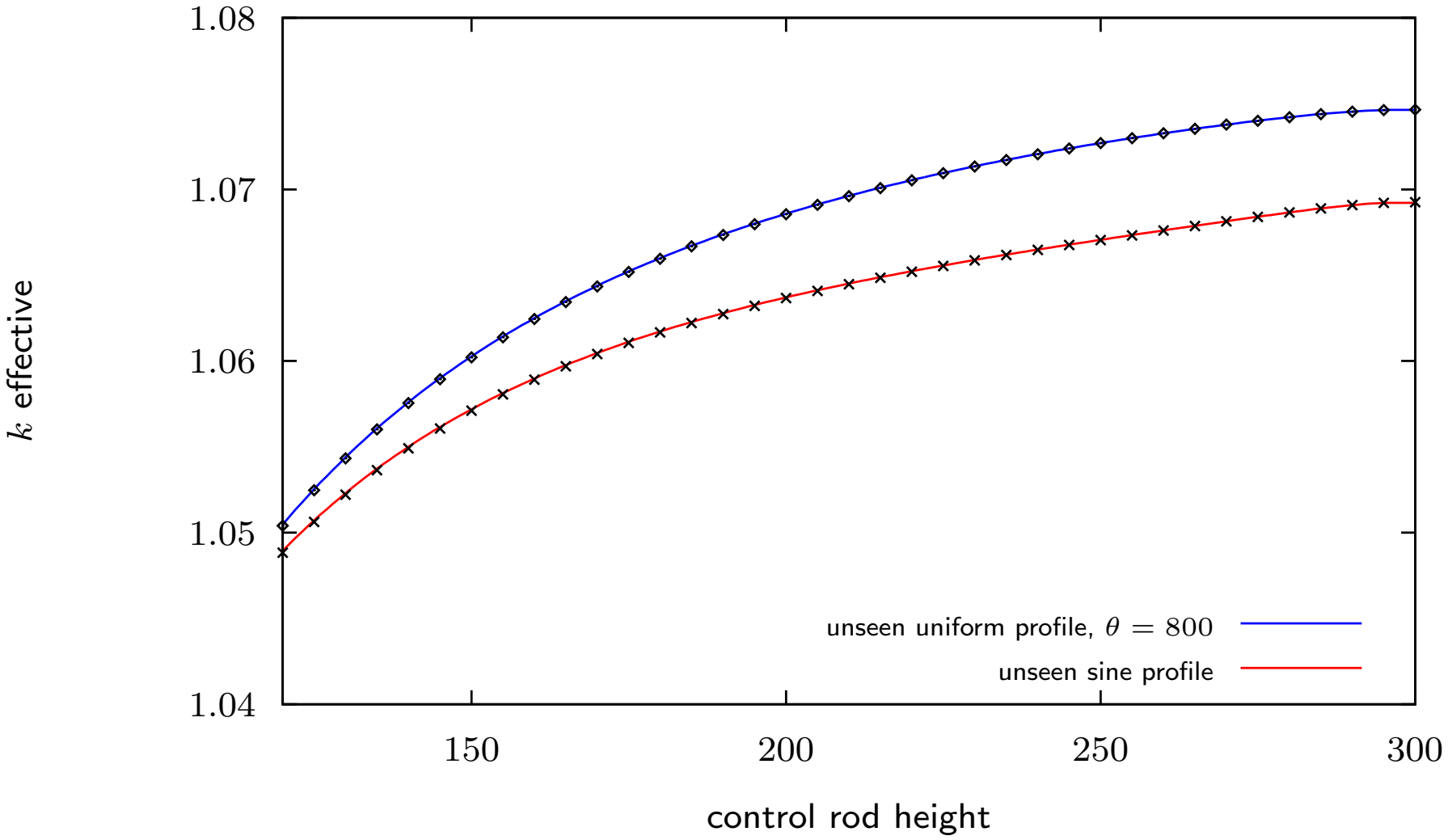

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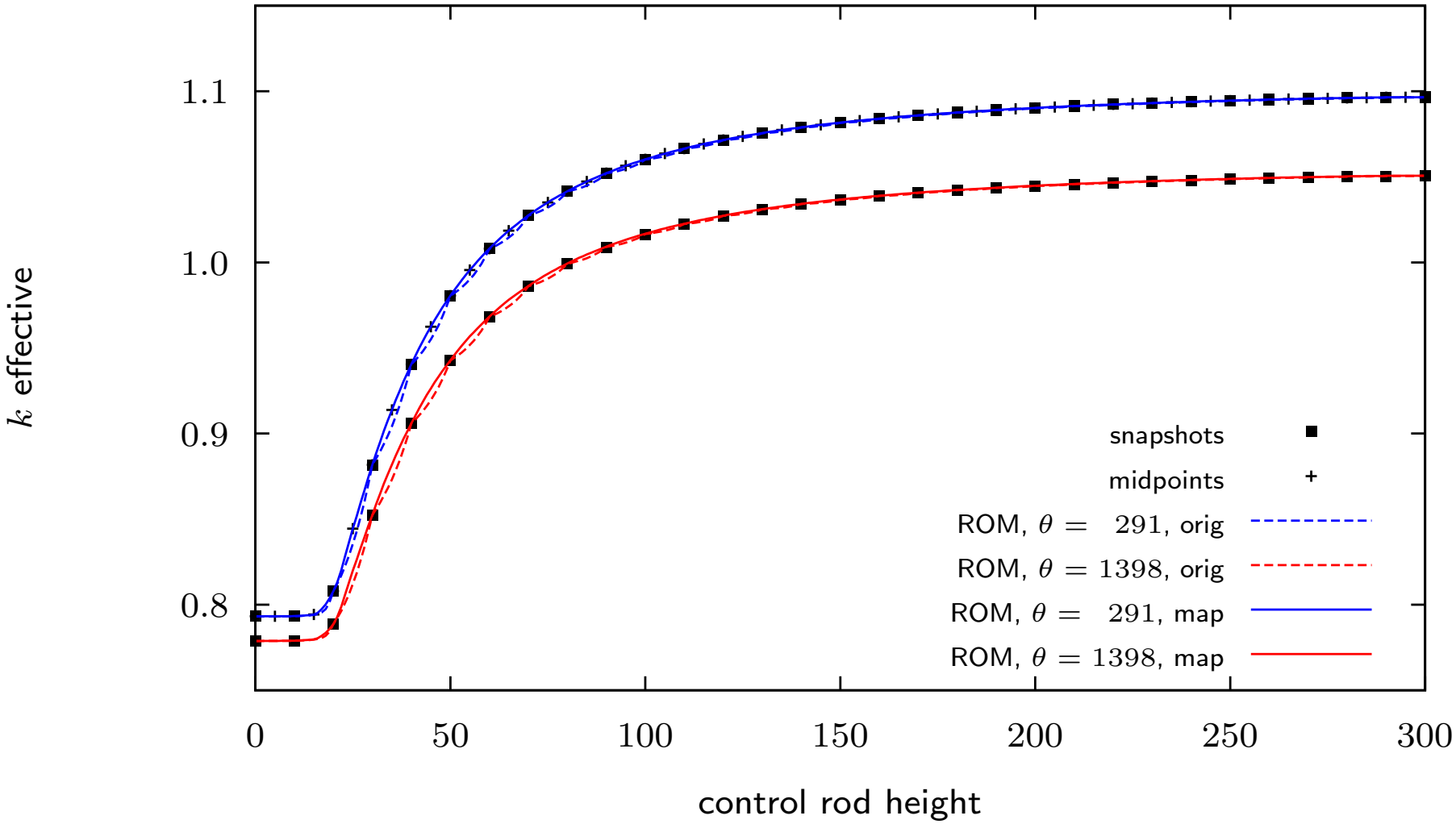
Values of k effective of the snapshots and ROM - local method



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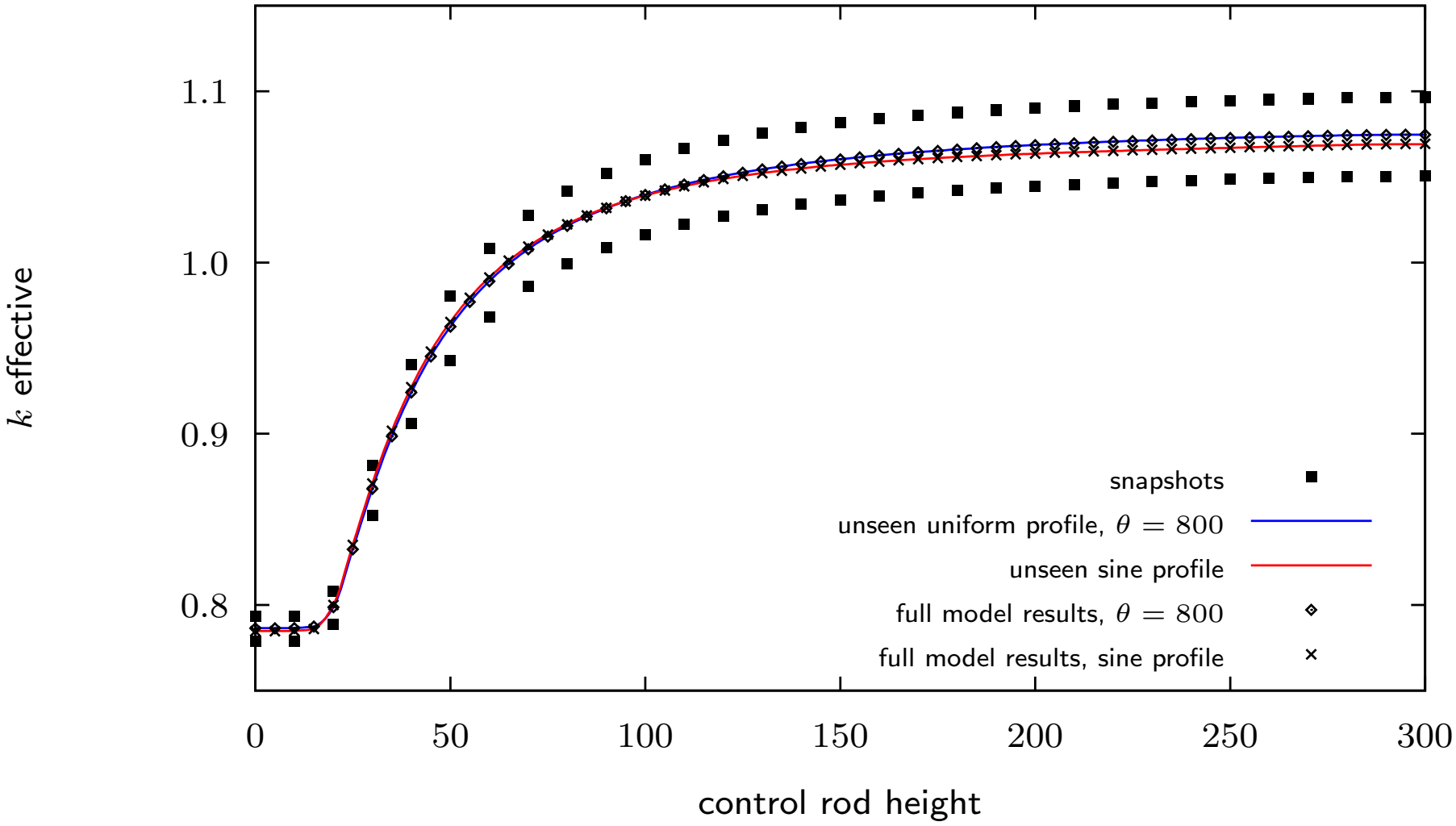

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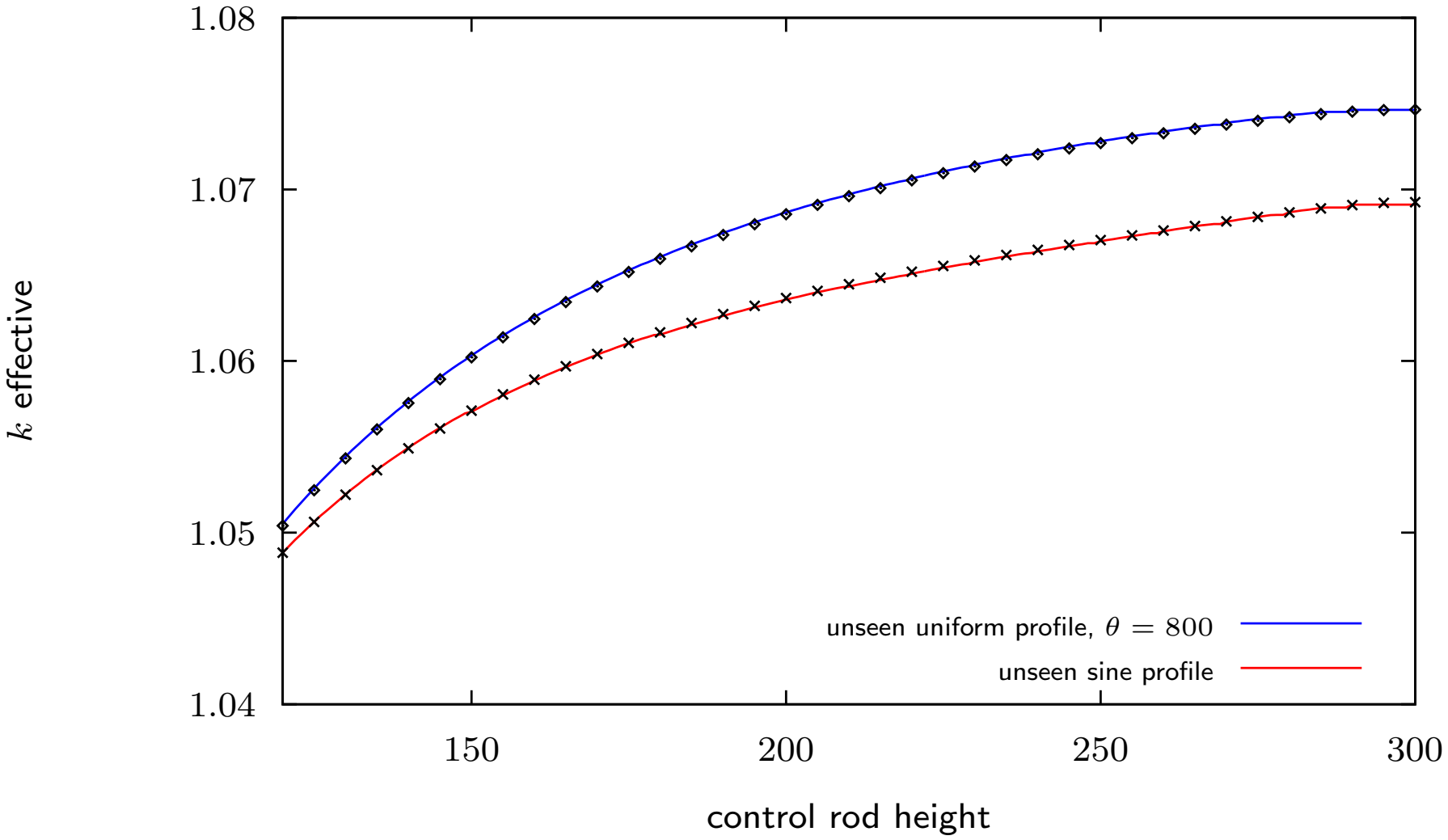

Results - Homogenised fuel assembly

Difference between k^{eff} of full model and ROM (tol 0.999995)

		$\max_i \lvert (k_i^{\text{full}} - k_i^{\text{rom}}) \rvert$	
	map	global	local
$\theta = 291$	L	9.7×10^{-3}	9.7×10^{-3}
	NL	7.8×10^{-5}	1.3×10^{-4}
$\theta = 800$	L	9.0×10^{-3}	9.0×10^{-3}
	NL	2.7×10^{-4}	2.3×10^{-4}
$\theta = 600 + 400\sin\left(\frac{\pi z}{300}\right)$	L	9.0×10^{-3}	9.1×10^{-3}
	NL	1.3×10^{-4}	1.6×10^{-4}

5. Results - Homogenised fuel assembly

Computation times

Calculation times

One full model (snapshot) calculation	458s
One ROM calculation	0.00254s.
Speed up	180,000

System sizes

Full order model:	39,528 by 39,528
ROM system:	42 by 42

Cost of forward model simulations

Snapshots:	28,396s (∼8 hours)
Midpoints:	13,740s (∼4 hours)
Total:	42,136s (∼12 hours)

6. Error approximation

Approximating the error in a ROM solution by using the (discretiesd) adjoint problem and by forming residuals of both the forward ROM and adjoint ROM problems:

forward ROM	$A\phi = \lambda B\phi$	$\mathrm{Res}(\phi) = A\phi - \lambda B\phi$
adjoint ROM	$A^{\mathsf{T}}\phi^* = \lambda^* B^{\mathsf{T}}\phi^*$	$\mathrm{Res}^*(\phi^*) = A^{\mathsf{T}}\phi^* - \lambda^* B^{\mathsf{T}}\phi^*$

For the discretised problem above, the error can be approximated by two expressions

$$(\delta\phi)^{\mathsf{T}}\,\mathrm{Res}^*(\phi^*) \qquad \text{or} \qquad (\delta\phi^*)^{\mathsf{T}}\,\mathrm{Res}(\phi)$$

where $\delta\phi$ is the error in the solution, $\delta\phi^*$ the error in the adjoint.

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6. Error approximation

How to approximate the error in the solution? By setting a certain number of the smallest coefficients to be zero.

$$\delta\phi \approx \phi_{[1,n]} - \phi_{[1,n-2]} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_{n-2} \\ \phi_{n-1} \\ \phi_n \end{Bmatrix} - \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_{n-2} \\ 0 \\ 0 \end{Bmatrix}$$

How to calculate the residual?

$$\mathrm{Res}(\phi_{[1,n-2]}) = A\phi_{[1,n-2]} - \lambda B\phi_{[1,n-2]}$$

Conclusions and further work

So far the reduced order models have been able to capture the behaviour of the neutronics of the full model extremely well.

Further work will include :

- determining confidence in ROM solutions through error measures
- more detailed modelling of the neutronics
  - resolving the fuel and clad, moderator and control rods separately
- modelling transient behaviour with temperature feedback by coupling neutronics with thermal hydraulics
- moving from assembly level to whole core models
- high fidelity, rapid, reduced order model simulator

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A couple of references

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Affine decomposition

An affine decomposition of the discretised operator A:

$$A = \sum_{i=1}^{n} \mu_i \tilde{A}_i \tag{8}$$

where $n \ll$ nDoF.

Consider the absorbtion term from the one group diffusion equation $\Sigma^a(x)\phi(x)$. Expanding $\phi(x) = N(x)\hat{\phi}$ and the discretisation is

$$\int_\Omega \Sigma^a(x) N^\mathsf{T}(x) N(x)\, dx\, \hat{\phi}\ . \tag{9}$$

Affine decomposition

Suppose we have constant temperature profiles imposed over the assembly, then Σ^a only varies according to the material, so we can write

$$\left(\Sigma^a_{\text{Fuel Reg}} \int_{\text{Fuel Reg}} N^{\mathsf{T}}(x) N(x)\, dx + \Sigma^a_{\text{CRod Reg}} \int_{\text{CRod Reg}} N^{\mathsf{T}}(x) N(x)\, dx\right) \hat{\phi}$$

$$\equiv \left(\Sigma^a_{\text{Fuel Reg}} A_{\text{Fuel Reg}} + \Sigma^a_{\text{CRod Reg}} A_{\text{CRod Reg}}\right) \hat{\phi}$$