Rational Solutions of the Boussinesq Equation and Applications to Rogue Waves

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References

- **A Ankiewicz, A P Bassom, P A Clarkson and E Dowie**, "Conservation laws and integral relations for the Boussinesq equation", *Studies in Applied Mathematics*, **139** (2017) 104–128
- **P A Clarkson and E Dowie**, "Rational solutions of the Boussinesq equation and applications to rogue waves", *Transactions of Mathematics and its Applications*, DOI: 10.1093/imatrm/tnx003 (2017)

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2. Rational solutions of the **nonlinear Schrödinger equation**

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4. Conservation laws for the Boussinesq equation
5. Rational solutions of the **Kadomtsev-Petviashvili I equation**

$$(u_t + 6uu_x + u_{xxx})_x = 3u_{yy} \qquad \text{KPI}$$

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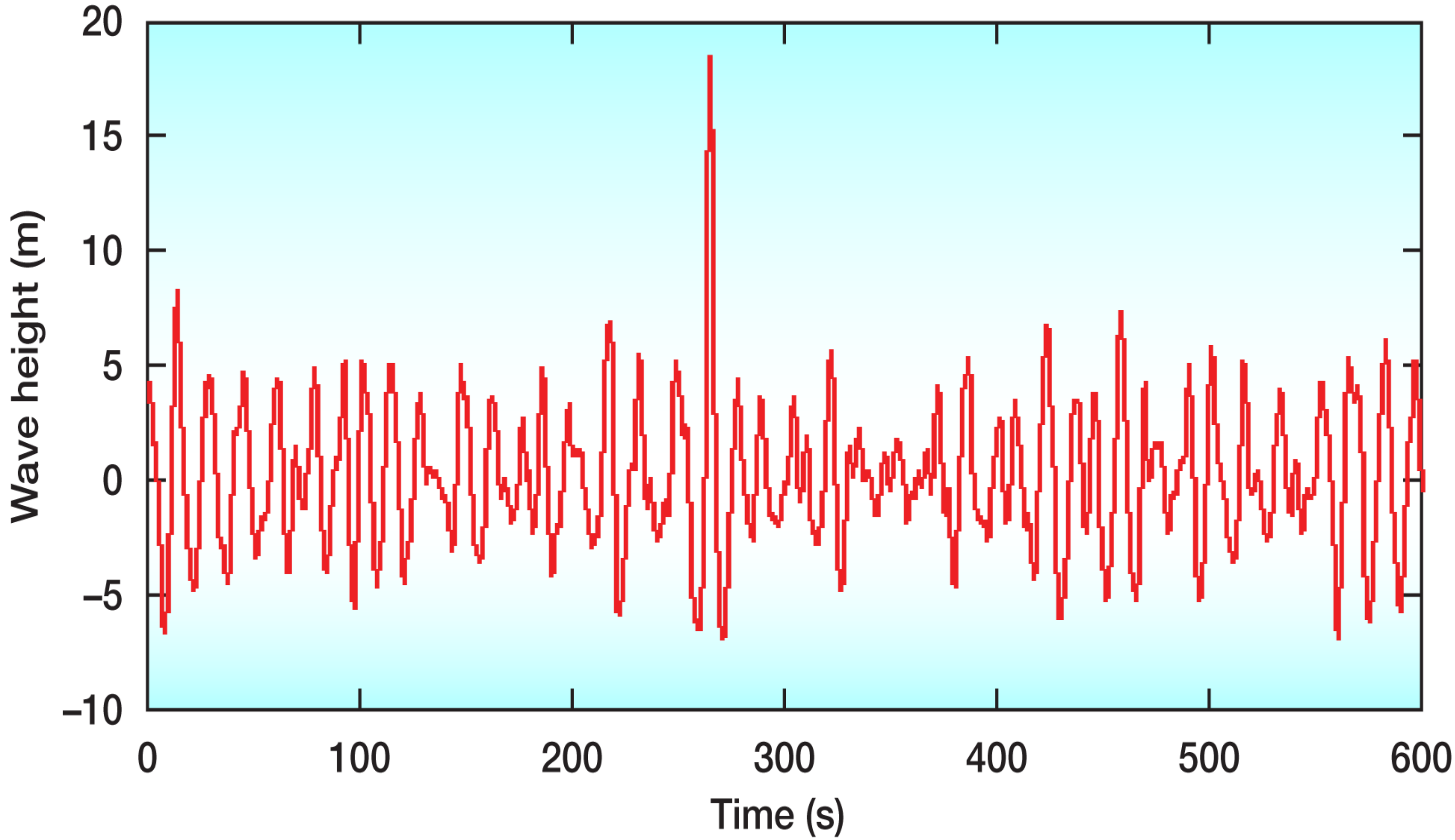
$$(u_t + 6uu_x + u_{xxx})_x - 3u_{yy} = 0 \qquad \text{KPI}$$

6. Conclusion and Open Problems

Rogue Waves

Rogue waves (or freak waves) are isolated structures with unusually high amplitude, such as the wave in the 1834 woodcut "*Fuji seen from the sea*" by Katsushika Hokusai.



Wave height measurement as a function of time showing the rogue wave observed on 1st January 1995 at the Draupner oil rig in the North Sea off the coast of Norway

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In recent years "rogue waves" have been observed in other contexts beyond the ocean

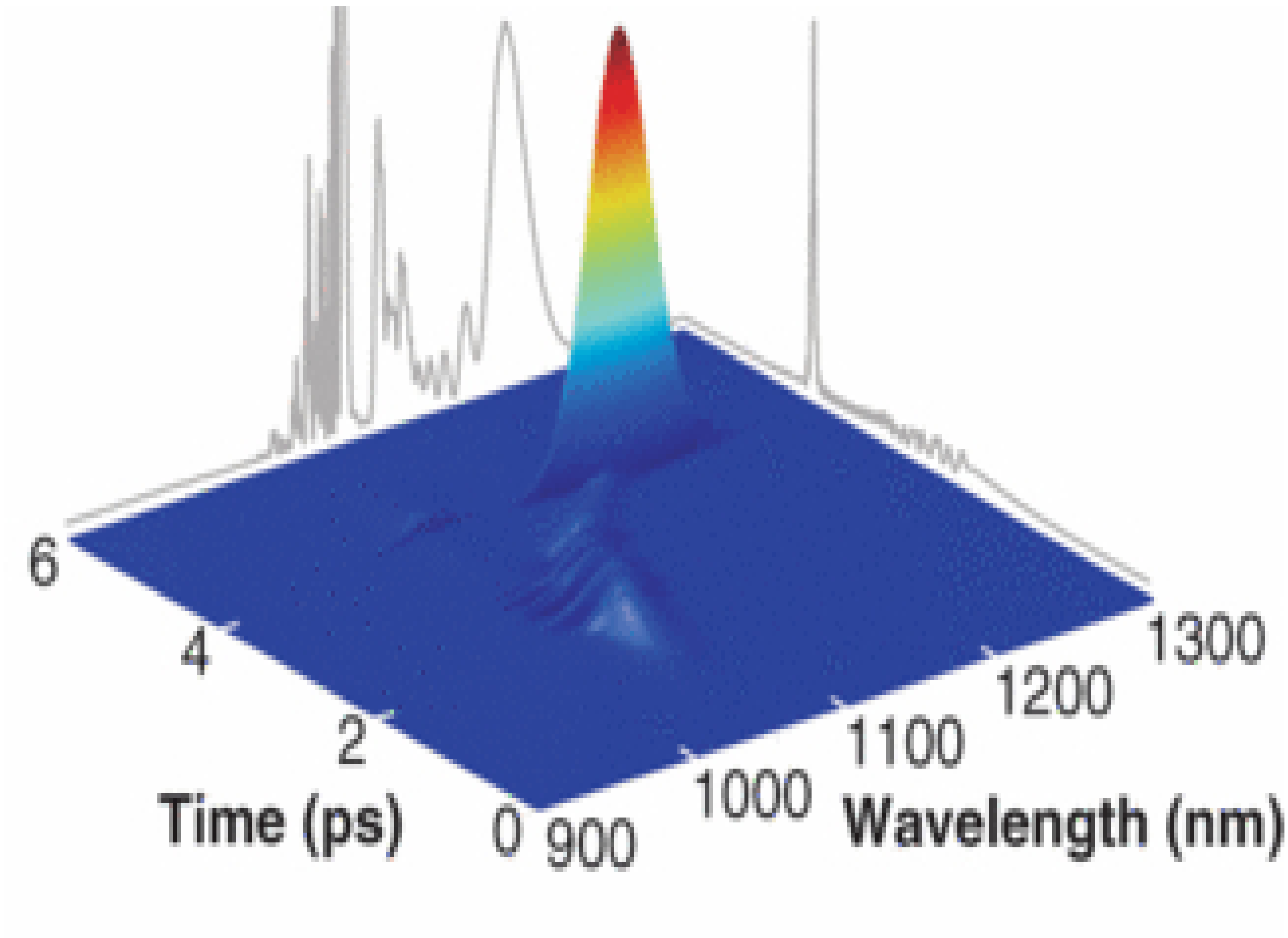
- Optical fibres (**Solli *et al.* [2007], Kilber *et al.* [2010]**).
  - *"How freak or rogue waves form in the ocean is not well understood, but new investigations suggest a mechanism for these waves that may also allow formation of high-intensity pulses in optical fibers"*
- Atmospheric waves (**Stenflo & Marklund [2010]**)
- Bose-Einstein condensates (**Bludov, Konotop & Akhmediev [2009]**)
- Waves in superfluids (**Ganshin *et al.* [2008]**)
- Plasma Physics (**Bailung, Sharma & Nakamura [2011]**)
- Finance (**Ivancevic [2009], Yan [2011]**).
  - The **Ivancevic option pricing model**

$$\mathrm{i}\frac{\partial \psi}{\partial t} + \tfrac{1}{2}\sigma\frac{\partial^2 \psi}{\partial S^2} + \beta|\psi|^2\psi = 0$$

where $\psi(S,t)$ the option price, S is the asset price, σ the volatility and β depends on the interest rate.

Time-wavelength profile of an optical rogue wave obtained from a short-time Fourier transform

(**Solli, Ropers, Koonath & Jalali**, *Nature* [2007])



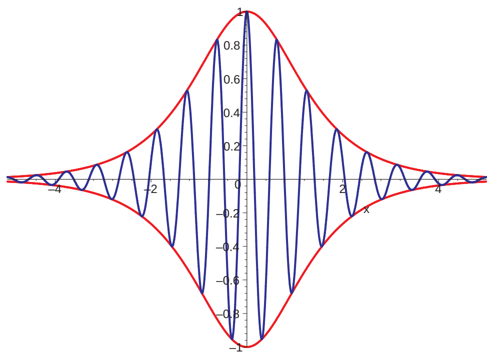
Rational Solutions of the Nonlinear Schrödinger Equation

$$\mathrm{i}\psi_t + \psi_{xx} \pm \tfrac{1}{2}|\psi|^2\psi = 0$$

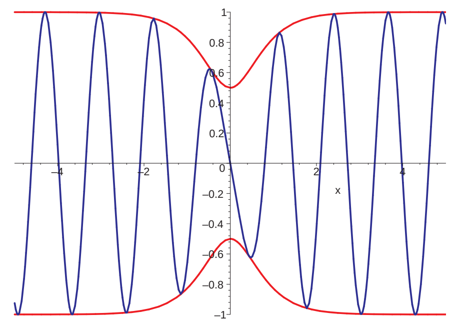
Nonlinear Schrödinger Equation

$$\mathrm{i}\psi_t + \psi_{xx} + \tfrac{1}{2}\sigma|\psi|^2\psi = 0, \qquad \sigma = \pm 1$$

- A **soliton equation** solvable by inverse scattering (**Zakharov & Shabat [1972]**); $\sigma = 1$ is "**focusing**" and $\sigma = -1$ is "**de-focusing**".



$\sigma = 1$: **Bright soliton**



$\sigma = -1$: **Dark soliton**

Nonlinear Schrödinger Equation

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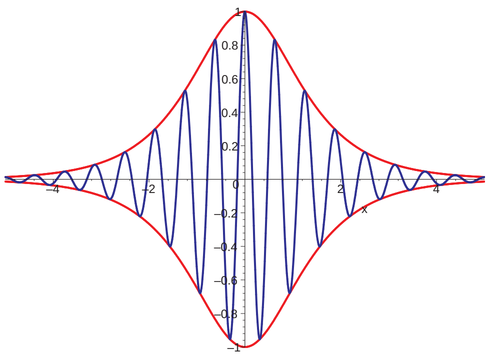

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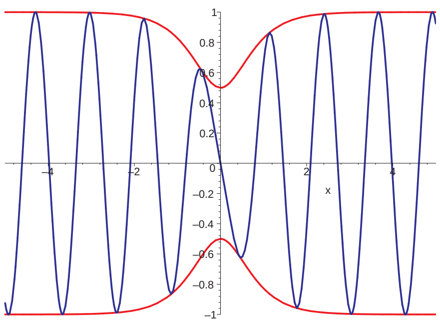

$\sigma = -1$: **Dark soliton**

- Arises in numerous physical applications including:
  - ▶ water waves (**Benney & Roukes [1969]; Zakharov [1968]**);
  - ▶ optical fibres (**Hasegawa & Tappert [1973]**);
  - ▶ plasmas (**Zakharov [1972]**);
  - ▶ ocean waves (**Peregrine [1983]**);
  - ▶ magnetostatic spin waves (**Kalinikos *et al*. [1997]; Xia *et al*. [1997]**).

Rational Solutions of the focusing NLS Equation

(Akhmediev, Ankiewicz & Soto-Crespo [2009])

(Akhmediev, Ankiewicz & PAC [2010])

Rational solutions of the **focusing NLS equation**

$$\mathrm{i}\psi_t + \psi_{xx} + \tfrac{1}{2}|\psi|^2\psi = 0$$

have the form

$$\psi_n(x,t) = \left\{1 - 4\frac{G_n(x,t) + \mathrm{i}tH_n(x,t)}{F_n(x,t)}\right\}\exp\left(\tfrac{1}{2}\mathrm{i}t\right)$$

where $F_n(x,t)$, $G_n(x,t)$ and $H_n(x,t)$ are polynomials in x and t with real coefficients, and $F_n(x,t)$ has no real zeros. The polynomials $F_n(x,t)$, $G_n(x,t)$ and $H_n(x,t)$ satisfy the Hirota equations

$$\begin{aligned}
&4(t\mathrm{D}_t + 1)H_n \bullet F_n + \mathrm{D}_x^2 F_n \bullet F_n - 4\mathrm{D}_x^2 F_n \bullet G_n = 0\\
&\mathrm{D}_t G_n \bullet F_n + t\mathrm{D}_x^2 H_n \bullet F_n = 0\\
&\mathrm{D}_x^2 F_n \bullet F_n = 8G_n^2 + 8t^2H_n^2 - 4F_nG_n
\end{aligned}$$

with D_x and D_t the Hirota operators

$$\mathrm{D}_x F \bullet G = \left(\frac{\mathrm{d}}{\mathrm{d}x_1} - \frac{\mathrm{d}}{\mathrm{d}x_2}\right) F(x_1)F(x_2)\bigg|_{x_1=x_2=x}$$

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Rational Solutions of the focusing NLS Equation

The first few rational solutions of the focusing NLS equation

$$\mathrm{i}\psi_t + \psi_{xx} + \tfrac{1}{2}|\psi|^2\psi = 0$$

have the form

$$\psi_0(x,t) = \exp\left(\tfrac{1}{2}\mathrm{i}t\right)$$

$$\psi_1(x,t) = \left\{1 - 4\,\frac{1+\mathrm{i}t}{x^2+t^2+1}\right\}\exp\left(\tfrac{1}{2}\mathrm{i}t\right)$$

$$\psi_2(x,t) = \left\{1 - 4\,\frac{G_2(x,t)+\mathrm{i}tH_2(x,t)}{F_2(x,t)}\right\}\exp\left(\tfrac{1}{2}\mathrm{i}t\right)$$

where

$$G_2(x,t) = 3\{x^4 + 6(t^2+1)x^2 + 5t^4 + 18t^2 - 3\}$$

$$H_2(x,t) = 3\{x^4 + 2(t^2-3)x^2 + (t^2+5)(t^2-3)\}$$

$$F_2(x,t) = x^6 + 3(t^2+1)x^4 + 3(t^2-3)^2x^2 + t^6 + 27t^4 + 99t^2 + 9$$

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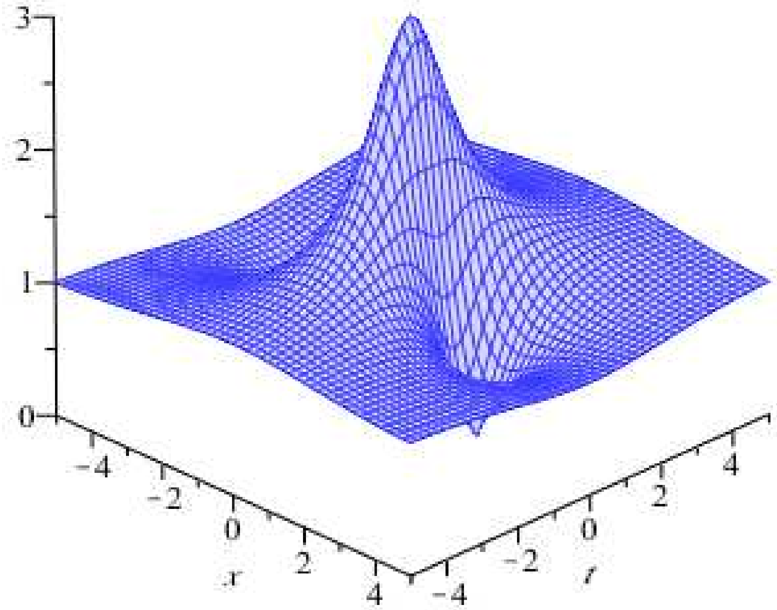
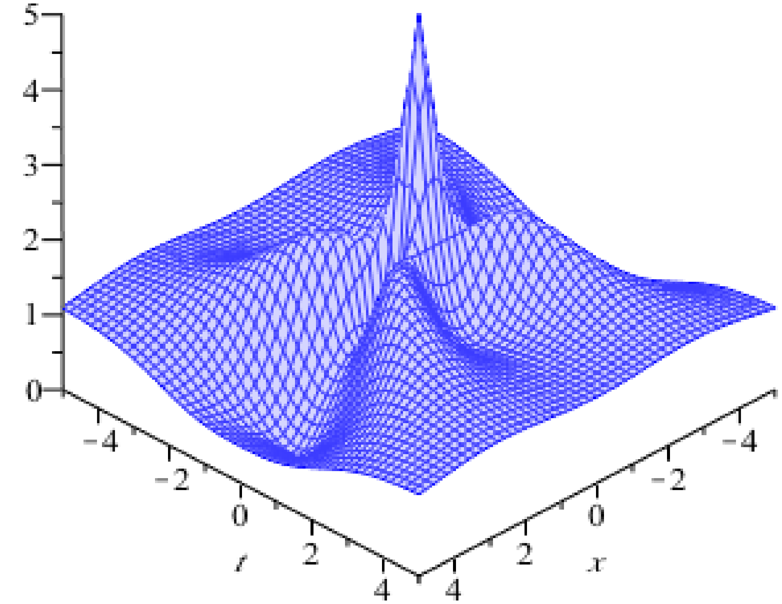
Remark

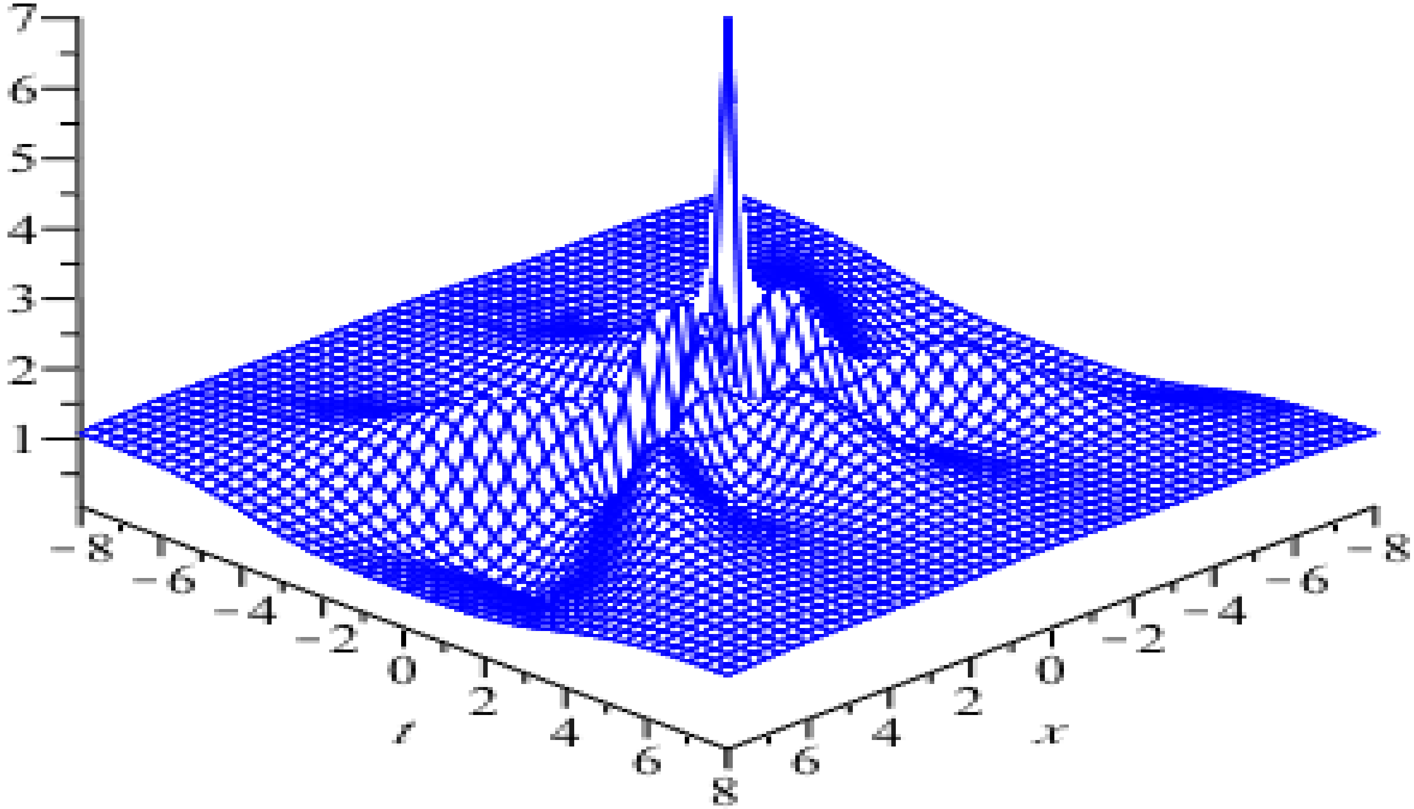
The solution $\psi_1(x,t)$ is the **Peregrine solution** (**Peregrine [1983]**).



$|\psi_1(x,t)|$



$|\psi_2(x,t)|$



$$|\psi_3(x,t)|$$

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Generalized Rational Solutions of the focusing NLS Equation

Dubard, Gaillard, Klein & Matveev [2010] show that the **focusing NLS equation**

$$\mathrm{i}\psi_t + \psi_{xx} + \tfrac{1}{2}|\psi|^2\psi = 0$$

has generalized rational solutions in the form

$$\widehat{\psi}_2(x,t;\alpha,\beta) = \left\{1 - 4\frac{\widehat{G}_2(x,t;\alpha,\beta) + \mathrm{i}\widehat{H}_2(x,t;\alpha,\beta)}{\widehat{F}_2(x,t;\alpha,\beta)}\right\}\exp\left(\tfrac{1}{2}\mathrm{i}t\right)$$

where

$$\widehat{G}_2(x,t;\alpha,\beta) = x^4 + 6(t^2+1)x^2 + 5t^4 + 18t^2 - 3 - 2\alpha t + 2\beta x$$

$$\widehat{H}_2(x,t;\alpha,\beta) = t\{x^4 + 2(t^2-3)x^2 + (t^2+5)(t^2-3)\} + \alpha(x^2 - t^2 + 1) + 2\beta t x$$

$$\begin{aligned}\widehat{F}_2(x,t;\alpha,\beta) = {} & x^6 + 3(t^2+1)x^4 + 3(t^2-3)^2x^2 + t^6 + 27t^4 + 99t^2 + 9 \\ & + 2\alpha t(3x^2 - t^2 - 9) + \alpha^2 - 2\beta x(x^2 - 3t^2 - 3) + \beta^2\end{aligned}$$

with α and β arbitrary constants — see also **Dubard & Matveev [2011, 2013]; Kedziora, Akhmediev & Ankiewicz [2011, 2012, 2013]**.

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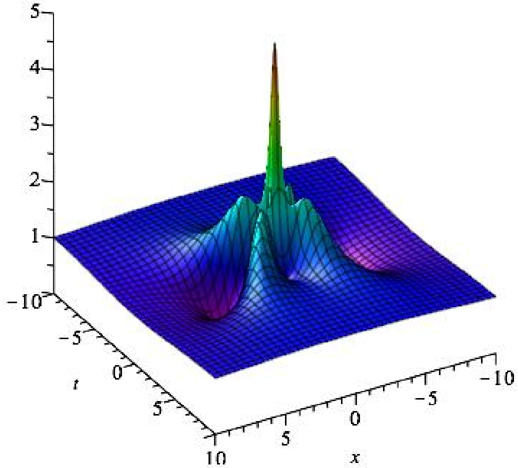
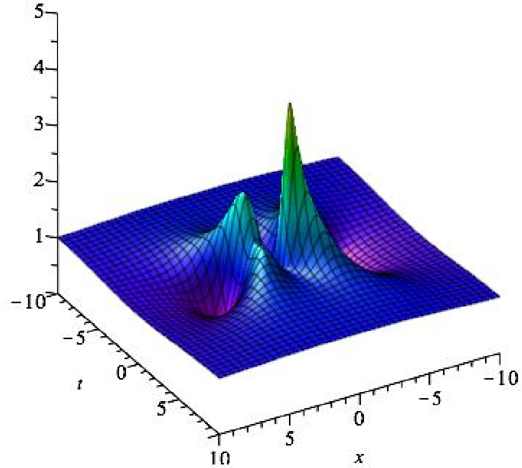
with α and β arbitrary constants — see also **Dubard & Matveev [2011, 2013]; Kedziora, Akhmediev & Ankiewicz [2011, 2012]**.

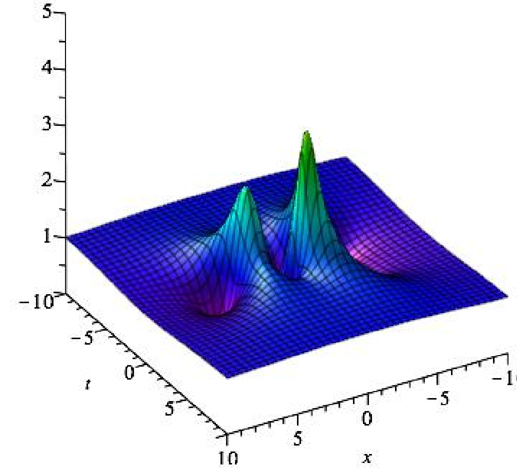
- These solutions have now been expressed in terms of Wronskians, see **Gaillard [2011, 2012, 2013, 2014, 2015, 2016]; Guo, Ling & Liu [2012]; Ohta & Yang [2012]**, ...



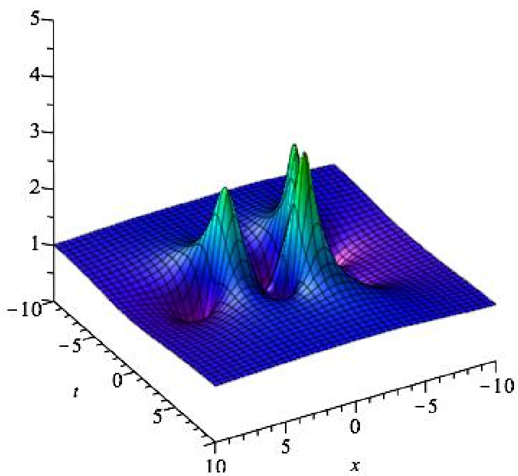
$\alpha = \beta = 0$



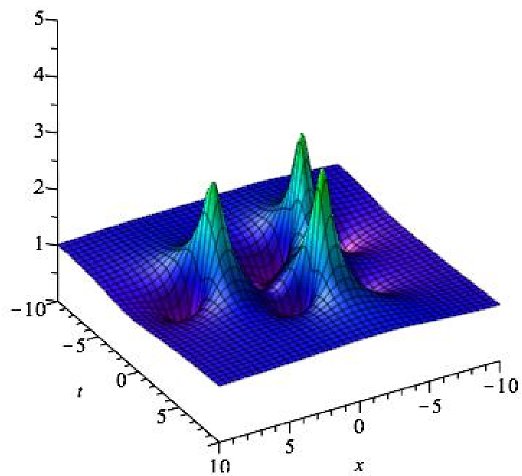
$\alpha = \beta = 5$



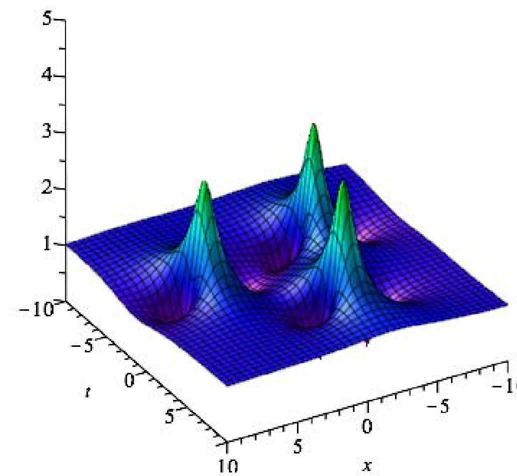
$\alpha = \beta = 10$



$\alpha = \beta = 20$



$\alpha = \beta = 50$



$\alpha = \beta = 100$

Rational Solutions of the Boussinesq Equation

$$u_{tt} + u_{xx} - (u^2)_{xx} - \tfrac{1}{3}u_{xxxx} = 0$$

Boussinesq Equation

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- A soliton equation solvable by inverse scattering (**Ablowitz & Haberman [1975], Zakharov [1974]**).

$$u(x,t) = 2\kappa^2 \operatorname{sech}^2\{\kappa(x-ct)\}, \qquad c = \pm\sqrt{\tfrac{4}{3}\kappa^2 - 1}$$

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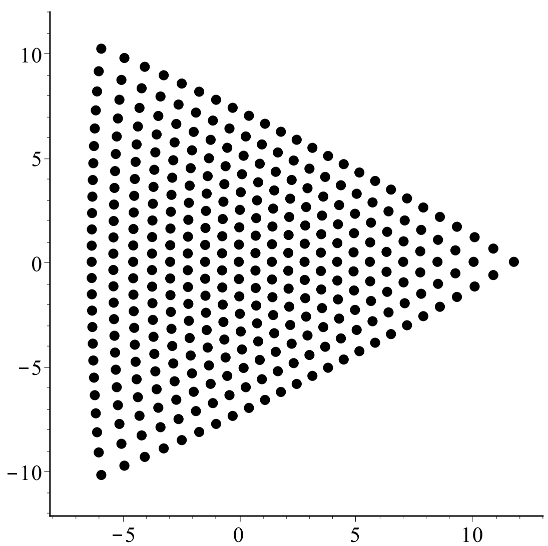
$$u(x,t) = 2\kappa^2 \operatorname{sech}^2\{\kappa(x-ct)\}, \qquad c = \pm\sqrt{\tfrac{4}{3}\kappa^2 - 1}$$

- Arises in several physical applications:
  - propagation of long waves in shallow water (**Boussinesq [1871]**, **Whitham [1974]**);
  - one-dimensional nonlinear lattice-waves (**Toda [1975]**);
  - the description of vibrations in a nonlinear string (**Zakharov [1974]**);
  - ion sound waves in a plasma (**Scott [1975]**).

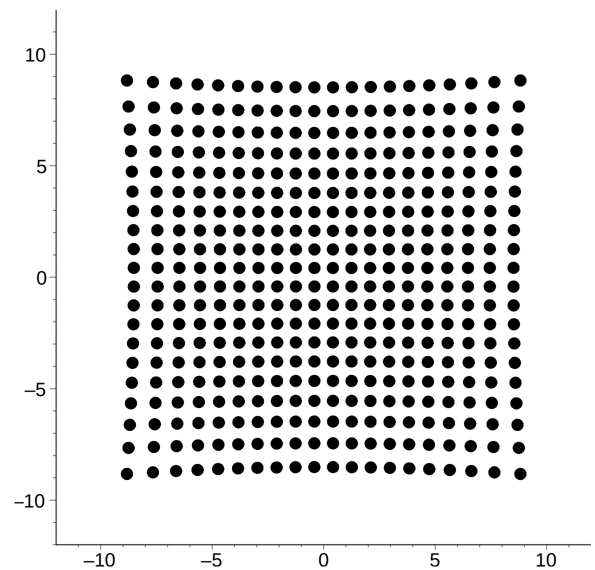
Rational Solutions of the Boussinesq Equation

$$u_{tt} + u_{xx} - (u^2)_{xx} - \tfrac{1}{3}u_{xxxx} = 0 \qquad (1)$$

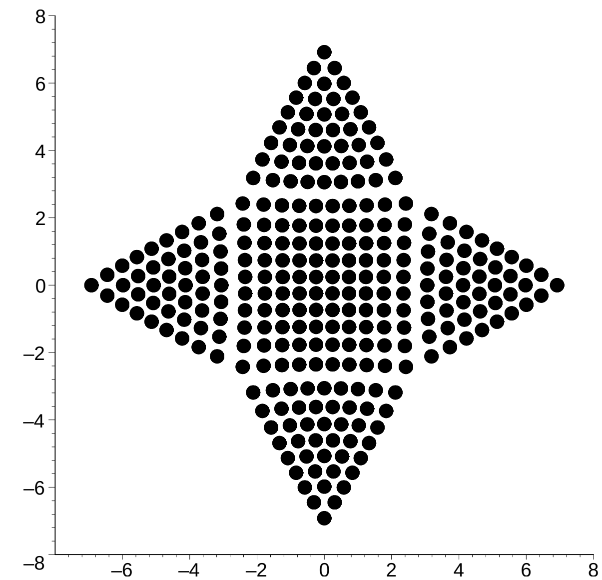
- The Boussinesq equation (1) has symmetry reductions which are solvable in terms of $\mathrm{P_{II}}$ and $\mathrm{P_{IV}}$. Hence rational solutions can be obtained in terms **Yablonskii–Vorob'ev polynomials**, which describe rational solutions of $\mathrm{P_{II}}$, and in terms of **generalised Hermite polynomials** and the **generalised Okamoto polynomials**, which describe rational solutions of $\mathrm{P_{IV}}$.



Yablonskii-Vorob'ev polynomials



generalised Hermite polynomials



generalised Okamoto polynomials

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Rational Solutions of the Boussinesq Equation

$$u_{tt} + u_{xx} - (u^2)_{xx} - \tfrac{1}{3}u_{xxxx} = 0 \qquad (1)$$

- The Boussinesq equation (1) has symmetry reductions which are solvable in terms of $\mathrm{P_{II}}$ and $\mathrm{P_{IV}}$. Hence rational solutions can be obtained in terms **Yablonskii–Vorob'ev polynomials**, which describe rational solutions of $\mathrm{P_{II}}$, and in terms of **generalised Hermite polynomials** and the **generalised Okamoto polynomials**, which describe rational solutions of $\mathrm{P_{IV}}$.
- There are also **generalised rational solutions** of the Boussinesq equation (1), which involve polynomials that are analogues of the **Burchnall-Chaundy/Adler-Moser polynomials** that arise in the description of rational solutions of the Korteweg-de Vries equation (**PAC [2008]**).

$$u_t + 6uu_x + u_{xxx} = 0$$

Rational Solutions of the Boussinesq Equation

$$u_{tt} + u_{xx} - (u^2)_{xx} - \tfrac{1}{3}u_{xxxx} = 0 \tag{1}$$

- The Boussinesq equation (1) has symmetry reductions which are solvable in terms of $\mathrm{P_{II}}$ and $\mathrm{P_{IV}}$. Hence rational solutions can be obtained in terms **Yablonskii–Vorob'ev polynomials**, which describe rational solutions of $\mathrm{P_{II}}$, and in terms of **generalised Hermite polynomials** and the **generalised Okamoto polynomials**, which describe rational solutions of $\mathrm{P_{IV}}$.
- There are also **generalised rational solutions** of the Boussinesq equation (1), which involve polynomials that are analogues of the **Burchnall-Chaundy/Adler-Moser polynomials** that arise in the description of rational solutions of the Korteweg-de Vries equation (**PAC [2008]**).

$$u_t + 6uu_x + u_{xxx} = 0$$

- However there are other rational solutions of the Boussinesq equation. **Ablowitz & Satsuma [1978]** obtained the rational solution

$$u(x,t) = \frac{4(1-x^2+t^2)}{(1+x^2+t^2)^2} = 2\frac{\partial^2}{\partial x^2}\ln(1+x^2+t^2)$$

by taking a long-wave limit of the two-soliton solution.

Rational Solutions of the Boussinesq Equation

$$u_{tt} + u_{xx} - (u^2)_{xx} - \tfrac{1}{3}u_{xxxx} = 0 \qquad (1)$$

Making the transformation

$$u(x,t) = 2\frac{\partial^2}{\partial x^2}\ln f(x,t)$$

yields the bilinear form

$$\left(\mathrm{D}_t^2 + \mathrm{D}_x^2 - \tfrac{1}{3}\mathrm{D}_x^4\right) f \bullet f = 0 \qquad (2)$$

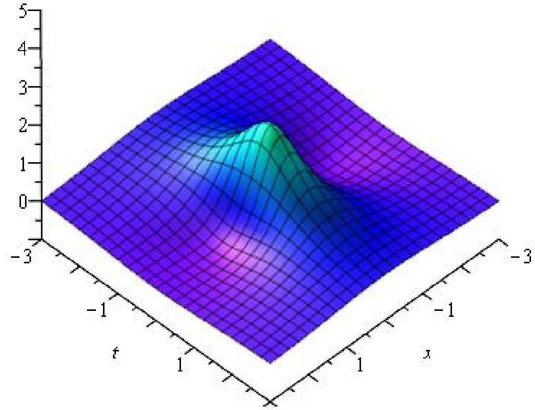
with D_x and D_t the Hirota operators. This has solutions $f_n(x,t)$ that are polynomials of degree $n(n+1)$ in both x and t, with $f_n(x,t) > 0$ for all $x, t \in \mathbb{R}$

$$f_1(x,t) = x^2 + t^2 + 1$$

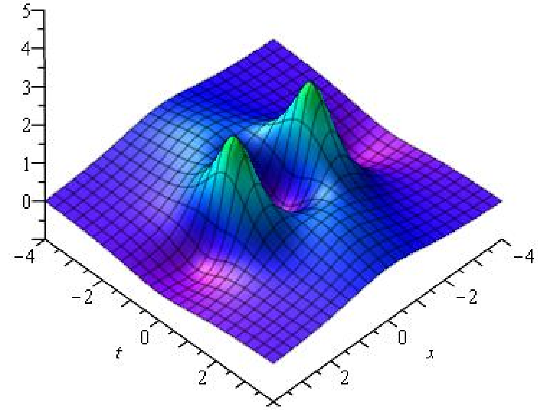
$$f_2(x,t) = x^6 + \left(3t^2 + \tfrac{25}{3}\right)x^4 + \left(3t^4 + 30t^2 - \tfrac{125}{9}\right)x^2 + t^6 + \tfrac{17}{3}t^4 + \tfrac{475}{9}t^2 + \tfrac{625}{9}$$

$$\begin{aligned} f_3(x,t) = x^{12} &+ \left(6t^2 + \tfrac{98}{3}\right)x^{10} + \left(15t^4 + 230t^2 + \tfrac{245}{3}\right)x^8 \\ &+ \left(20t^6 + \tfrac{1540}{3}t^4 + \tfrac{18620}{9}t^2 + \tfrac{75460}{81}\right)x^6 \\ &+ \left(15t^8 + \tfrac{1460}{3}t^6 + \tfrac{37450}{9}t^4 + \tfrac{24500}{3}t^2 - \tfrac{5187875}{243}\right)x^4 \\ &+ \left(6t^{10} + 190t^8 + \tfrac{35420}{9}t^6 - \tfrac{4900}{9}t^4 + \tfrac{188650}{27}t^2 + \tfrac{159786550}{729}\right)x^2 \\ &+ t^{12} + \tfrac{58}{3}t^{10} + \tfrac{1445}{3}t^8 + \tfrac{798980}{81}t^6 + \tfrac{16391725}{243}t^4 + \tfrac{300896750}{729}t^2 + \tfrac{878826025}{6561} \end{aligned}$$

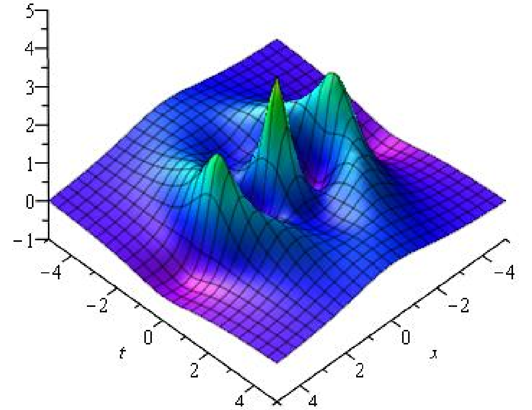
These polynomials appear in **Pelinovsky & Stepnyants [1992]**



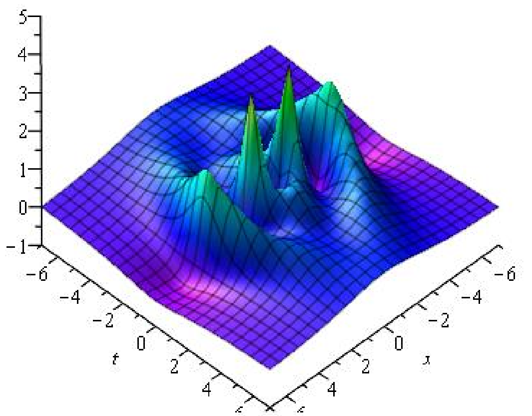
$u_1(x,t)$



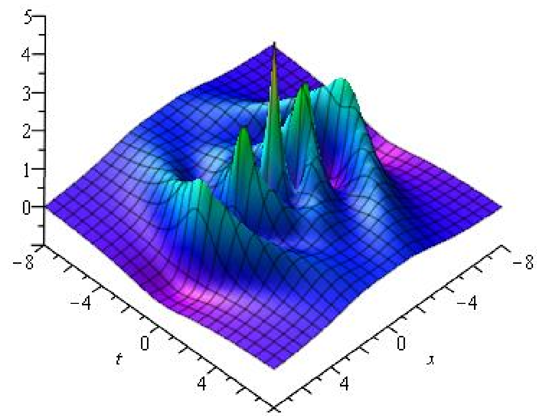
$u_2(x,t)$



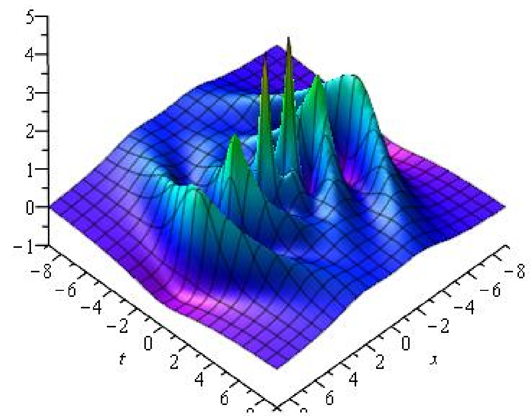
$u_3(x,t)$



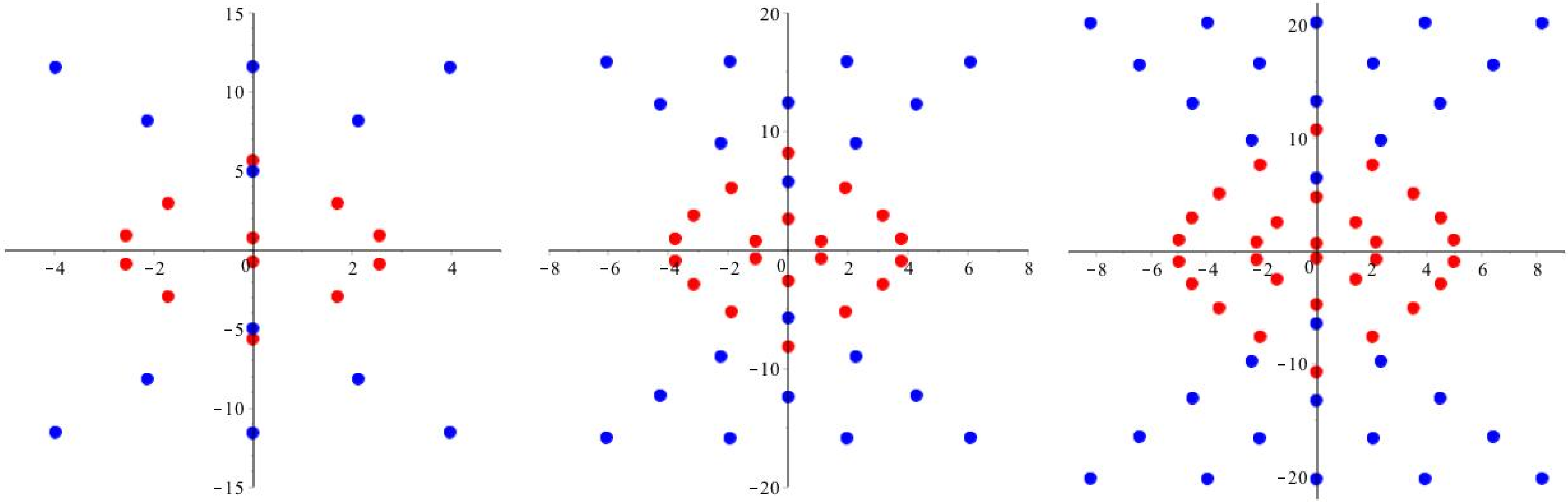
$u_4(x,t)$



$u_5(x,t)$

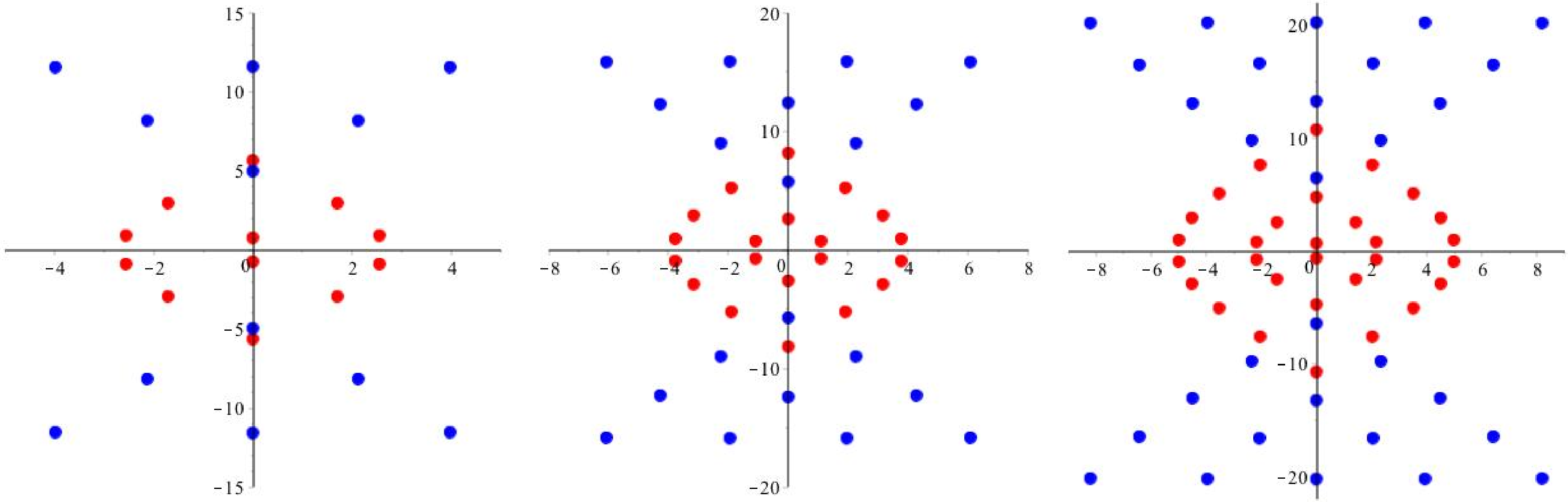


$u_6(x,t)$

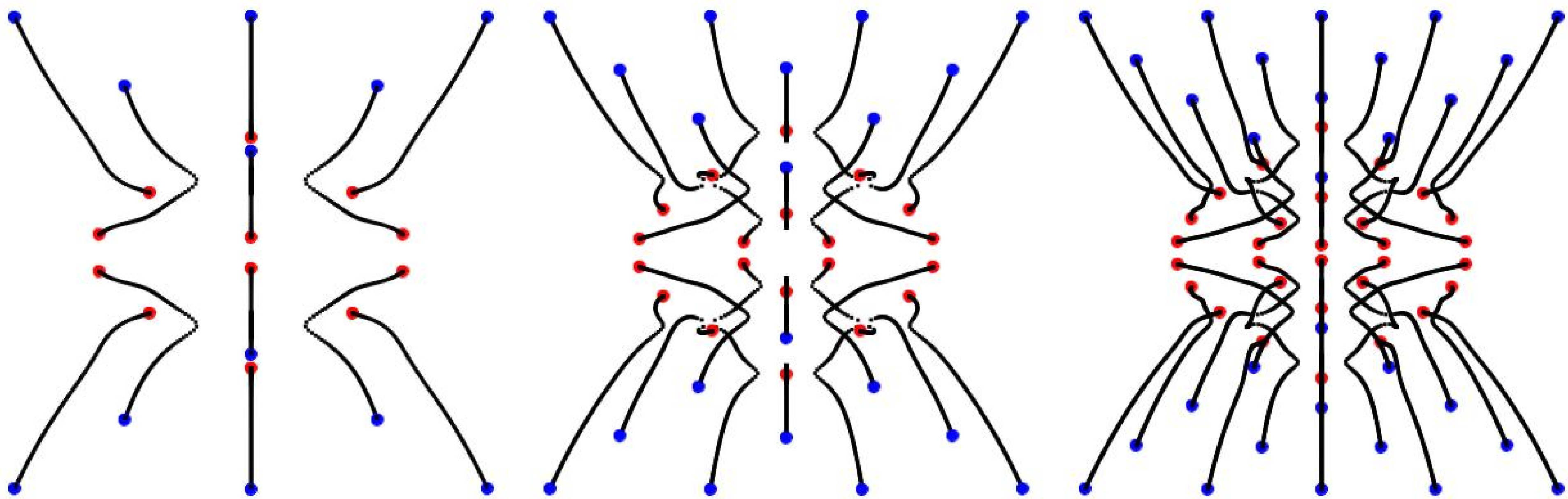


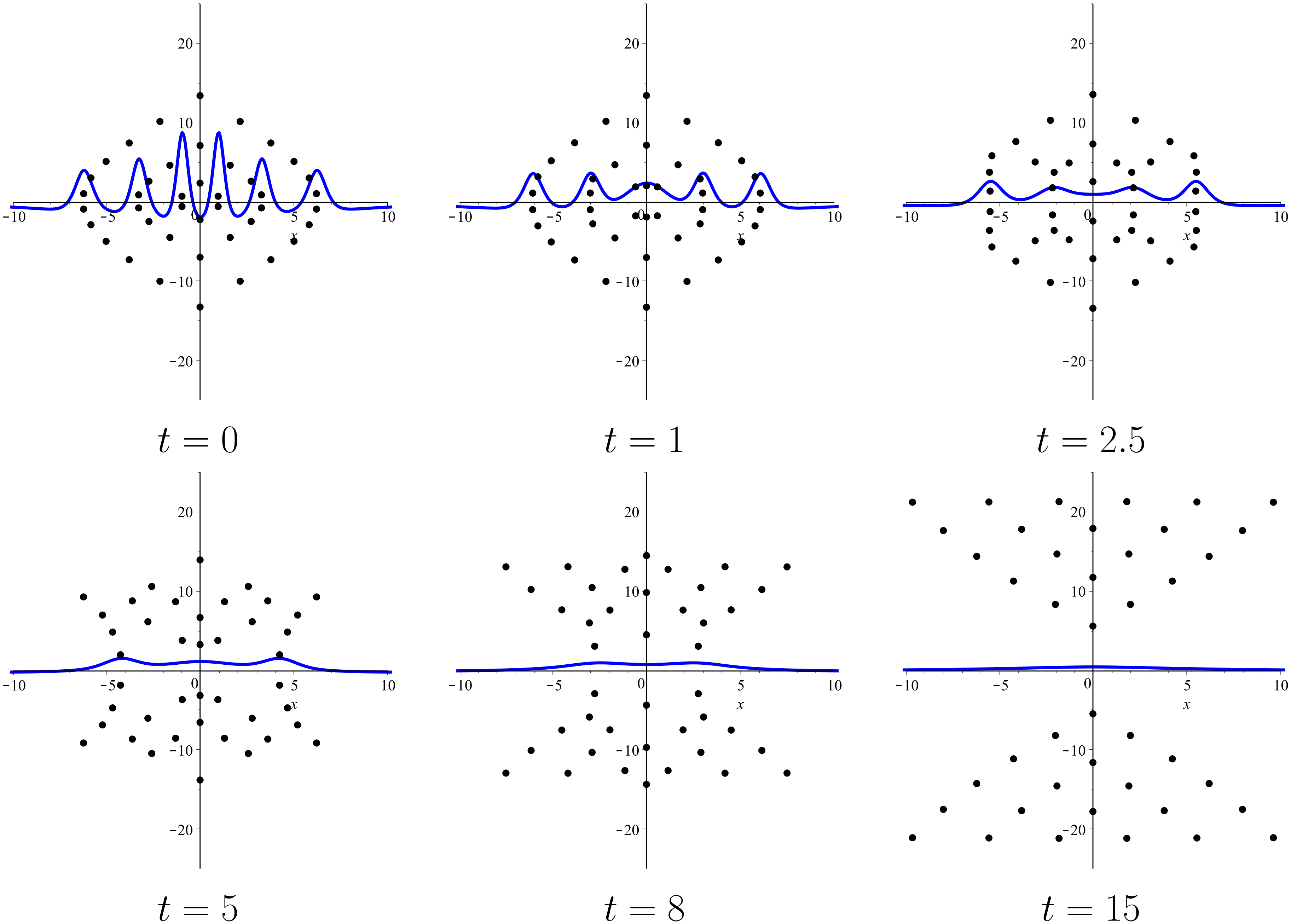
$F_3(x,t)$ $F_4(x,t)$ $F_5(x,t)$

Loci of the complex roots of the polynomials $F_n(x,t)$, for $3,4,5$, for $t=0$ (red) and $t=3n$ (blue), i.e. $t=9$ for $F_3(x,t)$, $t=12$ for $F_4(x,t)$ and $t=15$ for $F_5(x,t)$.



$F_3(x,t)$ $F_4(x,t)$ $F_5(x,t)$




$t = 0$
$t = 1$
$t = 2.5$
$t = 5$
$t = 8$
$t = 15$

Nonlinear Schrödinger equation

$$F_1(x,t) = x^2 + t^2 + 1$$

$$F_2(x,t) = \left(x^2+t^2\right)^3 + x^4 - 9(2t^2-3)x^2 + 27t^4 + 99t^2 + 9$$

$$\begin{aligned} F_3(x,t) = \left(x^2+t^2\right)^6 &+ 6x^{10} - 45(2t^2-3)x^8 - 180(t^4-3t^2-13)x^6 \\ &+ 15(4t^6 - 90t^4 + 900t^2 + 225)x^4 \\ &+ 6(45t^8 + 2250t^6 + 13050t^4 - 6075t^2 + 2025)x^2 \\ &+ 126t^{10} + 3735t^8 + 15300t^6 + 143775t^4 + 93150t^2 + 2025 \end{aligned}$$

Boussinesq equation

$$f_1(x,t) = x^2 + t^2 + 1$$

$$f_2(x,t) = \left(x^2+t^2\right)^3 + \tfrac{25}{3}x^4 + \left(30t^2 - \tfrac{125}{9}\right)x^2 + \tfrac{17}{3}t^4 + \tfrac{475}{9}t^2 + \tfrac{625}{9}$$

$$\begin{aligned} f_3(x,t) = \left(x^2+t^2\right)^6 &+ \tfrac{98}{3}x^{10} + \left(230t^2 + \tfrac{245}{3}\right)x^8 + \left(\tfrac{1540}{3}t^4 + \tfrac{18620}{9}t^2 + \tfrac{75460}{81}\right)x^6 \\ &+ \left(\tfrac{1460}{3}t^6 + \tfrac{37450}{9}t^4 + \tfrac{24500}{3}t^2 - \tfrac{5187875}{243}\right)x^4 \\ &+ \left(190t^8 + \tfrac{35420}{9}t^6 - \tfrac{4900}{9}t^4 + \tfrac{188650}{27}t^2 + \tfrac{159786550}{729}\right)x^2 \\ &+ \tfrac{58}{3}t^{10} + \tfrac{1445}{3}t^8 + \tfrac{798980}{81}t^6 + \tfrac{16391725}{243}t^4 + \tfrac{300896750}{729}t^2 + \tfrac{878826025}{6561} \end{aligned}$$

Generalised Rational Solution of the Boussinesq Equation

The Boussinesq equation

$$u_{tt} + u_{xx} - (u^2)_{xx} - \tfrac{1}{3}u_{xxxx} = 0$$

also has the **generalised rational solution**

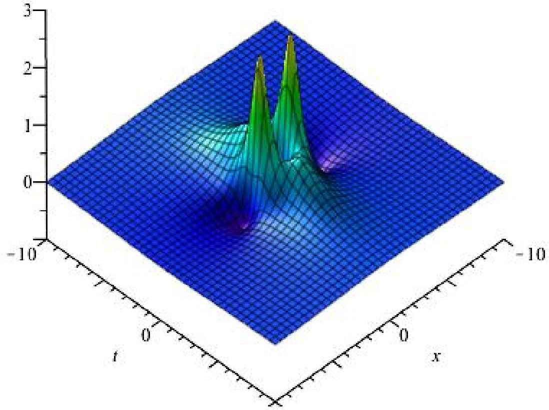
$$\widetilde{u}_2(x,t;\alpha,\beta) = 2\frac{\partial^2}{\partial x^2}\ln \widetilde{f}_2(x,t;\alpha,\beta)$$

with

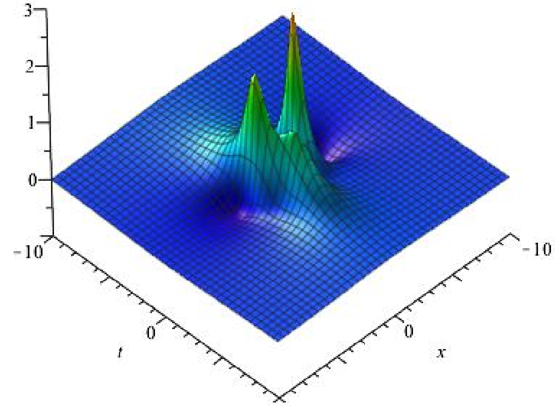
$$\begin{aligned}\widetilde{f}_2(x,t;\alpha,\beta) &= \left(x^2+t^2\right)^3 + \tfrac{25}{3}x^4 + \left(30t^2 - \tfrac{125}{9}\right)x^2 + \tfrac{17}{3}t^4 + \tfrac{475}{9}t^2 + \tfrac{625}{9}\\ &\qquad + 2\alpha t\left(3x^2 - t^2 + \tfrac{5}{3}\right) + 2\beta x\left(x^2 - 3t^2 - \tfrac{1}{3}\right) + \alpha^2 + \beta^2\\ &= f_2(x,t) + 2\alpha t\left(3x^2 - t^2 + \tfrac{5}{3}\right) + 2\beta x\left(x^2 - 3t^2 - \tfrac{1}{3}\right) + \alpha^2 + \beta^2\end{aligned}$$

with α and β arbitrary constants.

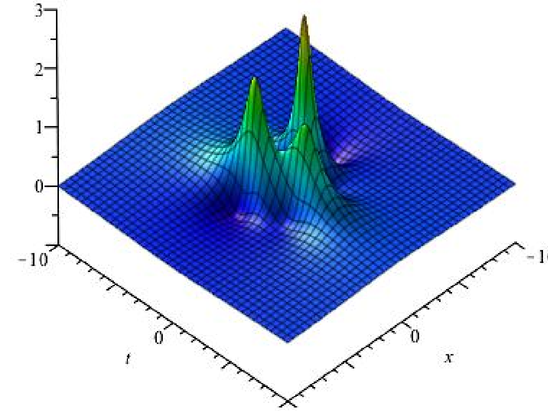
$$\widetilde{u}_2(x,t;\alpha,\beta)$$



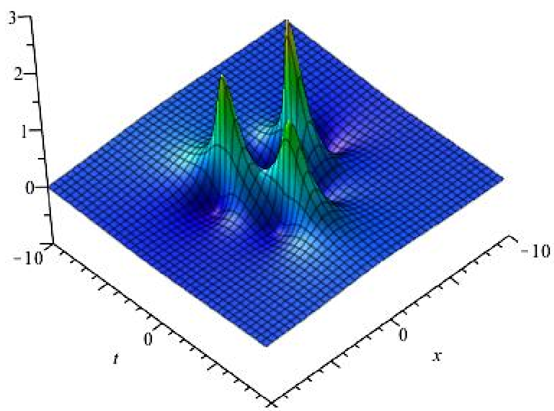
$\alpha = \beta = 0$



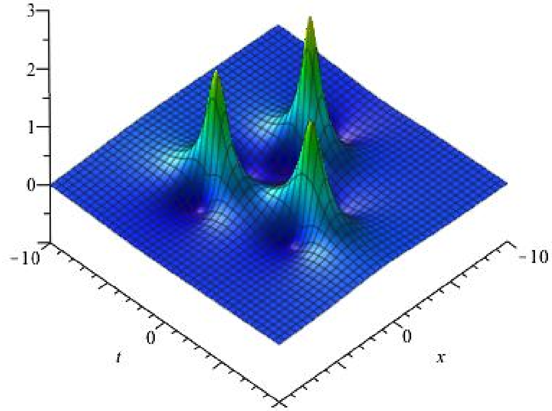
$\alpha = \beta = 5$



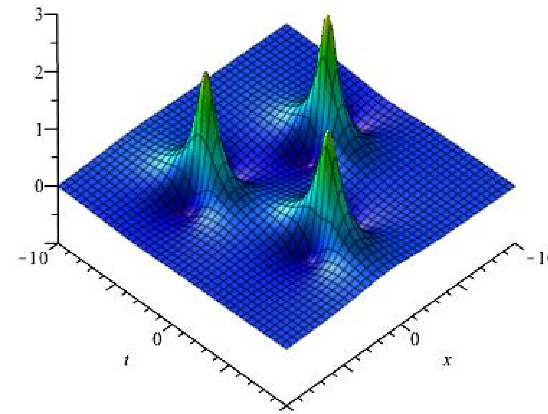
$\alpha = \beta = 10$



$\alpha = \beta = 20$

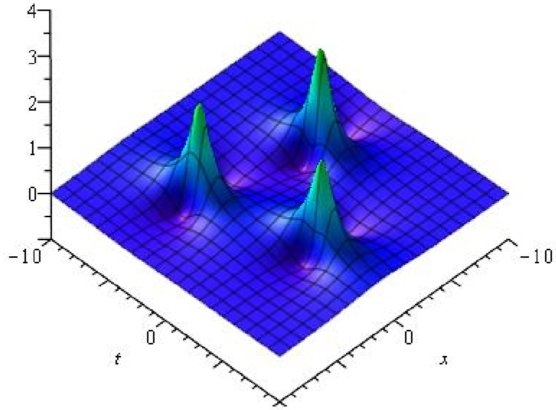


$\alpha = \beta = 50$

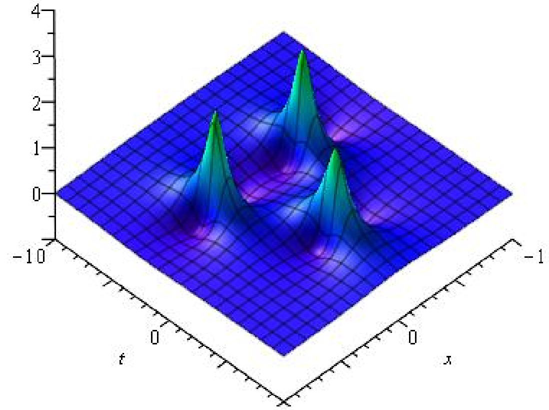


$\alpha = \beta = 100$

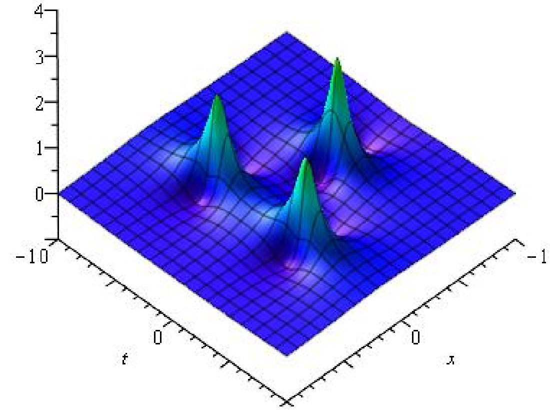
$$\widetilde{u}_2(x,t;\alpha,\beta)$$



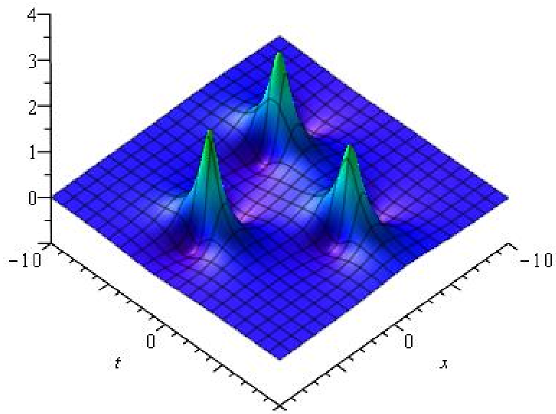
$\alpha = \beta = 100$



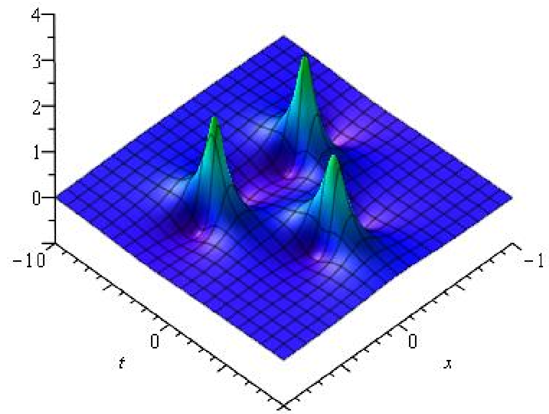
$\alpha = 100, \beta = 0$



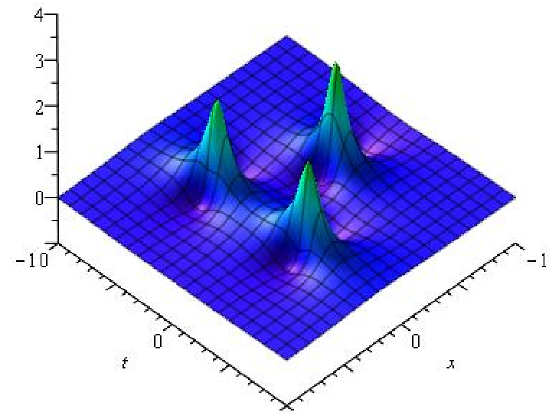
$\alpha = 0, \beta = 100$



$\alpha = 100, \beta = -100$



$\alpha = 100, \beta = 10$



$\alpha = 10, \beta = 100$

The next generalised rational solution is

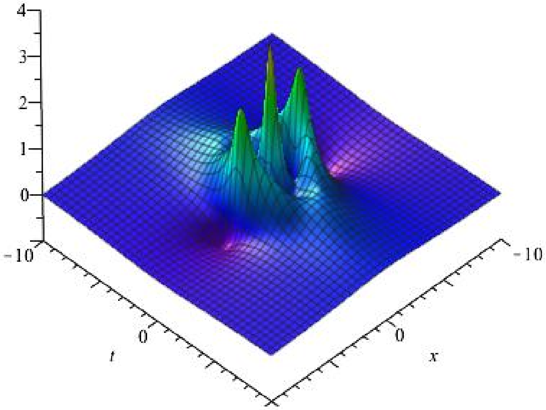
$$\widetilde{u}_3(x,t;\alpha,\beta) = 2\frac{\partial^2}{\partial x^2}\ln \widetilde{f}_3(x,t;\alpha,\beta)$$

with

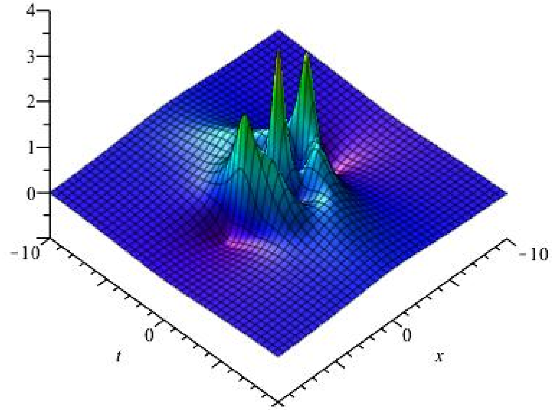
$$\begin{aligned}
\widetilde{f}_3(x,t;\alpha,\beta) &= f_3(x,t) + 2\alpha t p_2(x,t) + 2\beta x q_2(x,t) + (\alpha^2+\beta^2) f_1(x,t) \\
&= \left(x^2+t^2\right)^6 + \tfrac{98}{3}x^{10} + \left(230t^2 + \tfrac{245}{3}\right)x^8 \\
&\quad + \left(\tfrac{1540}{3}t^4 + \tfrac{18620}{9}t^2 + \tfrac{75460}{81}\right)x^6 \\
&\quad + \left(\tfrac{1460}{3}t^6 + \tfrac{37450}{9}t^4 + \tfrac{24500}{3}t^2 - \tfrac{5187875}{243}\right)x^4 \\
&\quad + \left(190t^8 + \tfrac{35420}{9}t^6 - \tfrac{4900}{9}t^4 + \tfrac{188650}{27}t^2 + \tfrac{159786550}{729}\right)x^2 \\
&\quad + \tfrac{58}{3}t^{10} + \tfrac{1445}{3}t^8 + \tfrac{798980}{81}t^6 + \tfrac{16391725}{243}t^4 + \tfrac{300896750}{729}t^2 + \tfrac{878826025}{6561} \\
&\quad + 2\alpha t\left\{t^6 - (9x^2+7)t^4 - (5x^4+190x^2+245)t^2 \right. \\
&\qquad\qquad \left. + 5x^6 + 105x^4 - 665x^2 + \tfrac{18865}{3}\right\} \\
&\quad + 2\beta x\left\{x^6 - (9t^2-13)x^4 - (5t^4+230t^2+245)x^2 \right. \\
&\qquad\qquad \left. + 5t^6 + 45t^4 + 535t^2 + \tfrac{12005}{3}\right\} \\
&\quad + (\alpha^2+\beta^2)(x^2+t^2+1)
\end{aligned}$$

with α and β arbitrary constants.

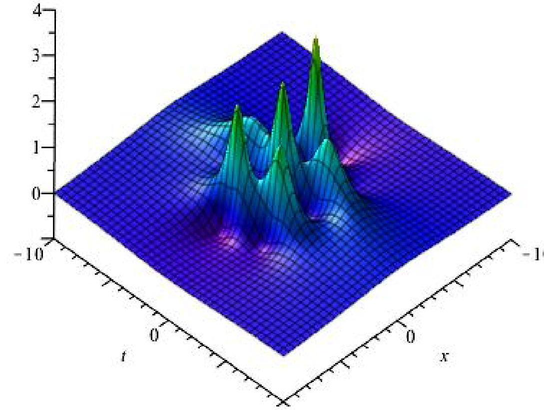
$$\widetilde{u}_3(x, t; \alpha, \beta)$$



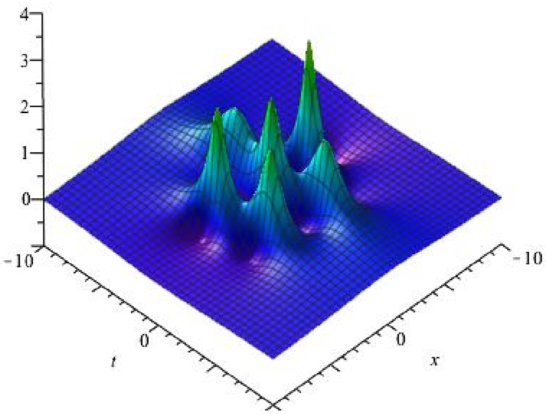
$\alpha = \beta = 0$



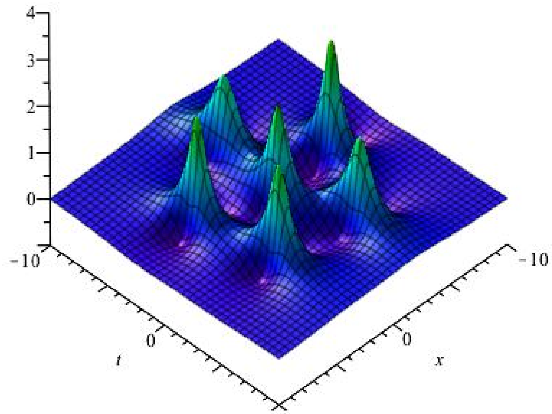
$\alpha = \beta = 100$



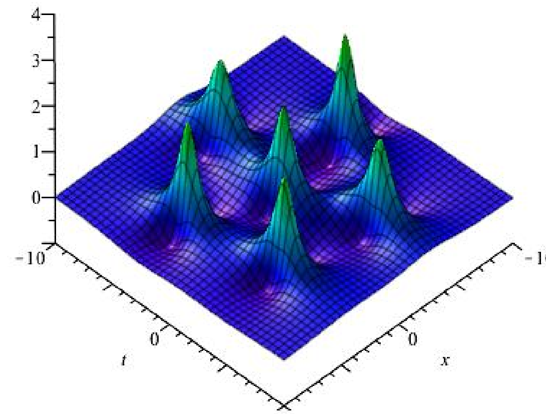
$\alpha = \beta = 500$



$\alpha = \beta = 1000$

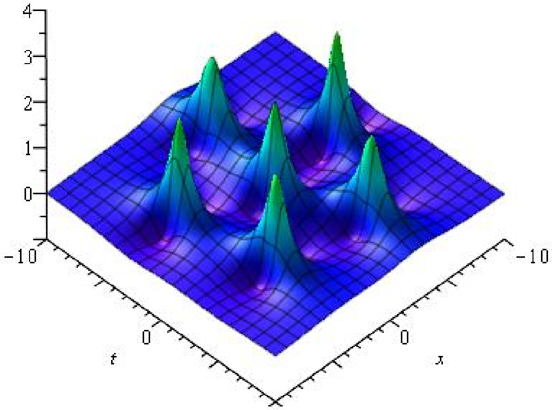


$\alpha = \beta = 5000$

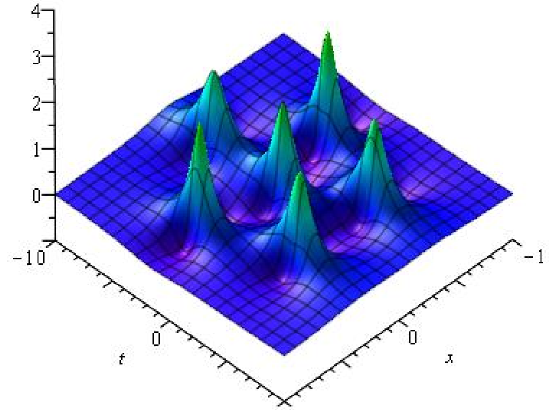


$\alpha = \beta = 10000$

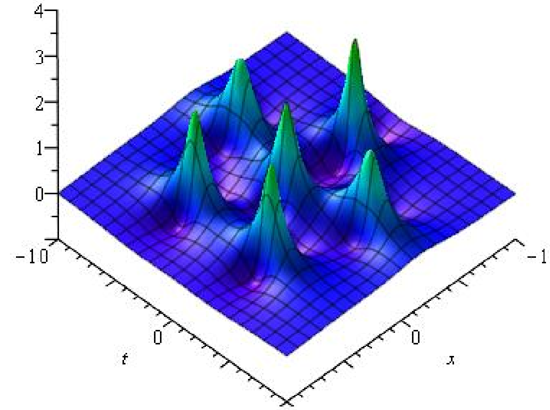
$$\widetilde{u}_3(x,t;\alpha,\beta)$$



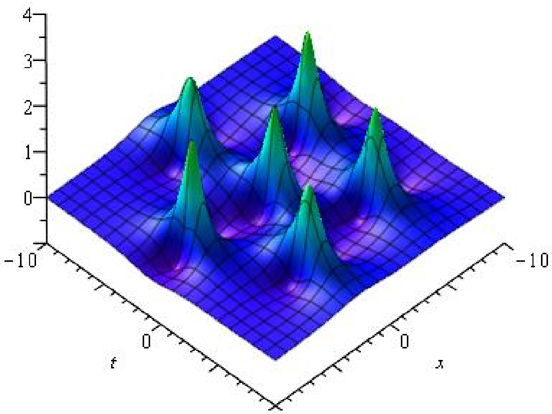
$\alpha = \beta = 10^4$



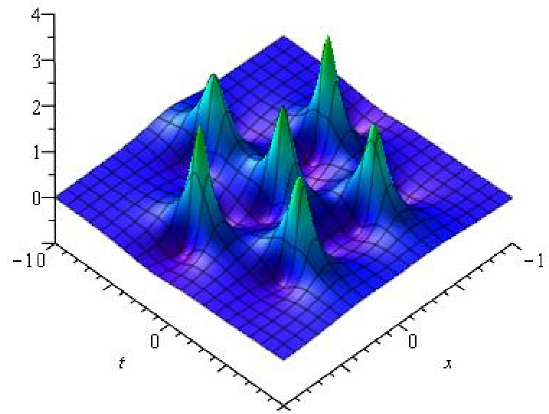
$\alpha = 10^4, \beta = 0$



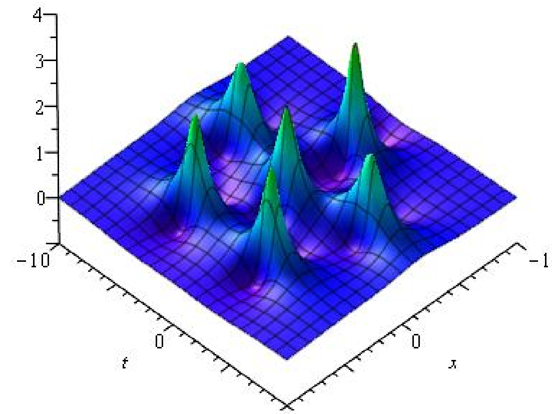
$\alpha = 0, \beta = 10^4$



$\alpha = 10^4, \beta = -10^4$

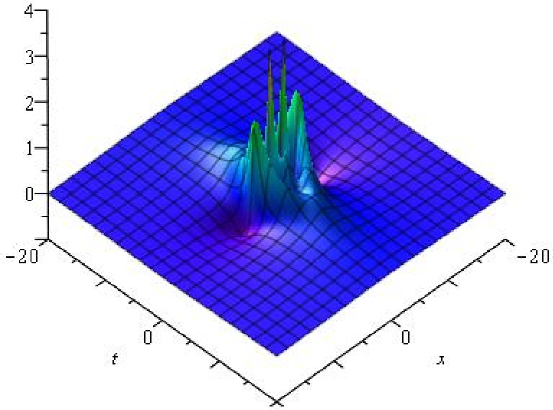


$\alpha = 10^4, \beta = 10^2$

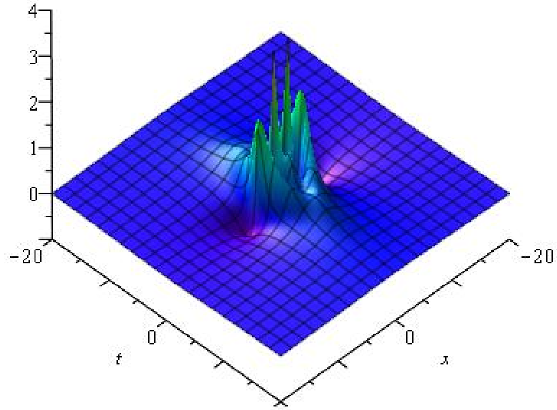


$\alpha = 10^2, \beta = 10^4$

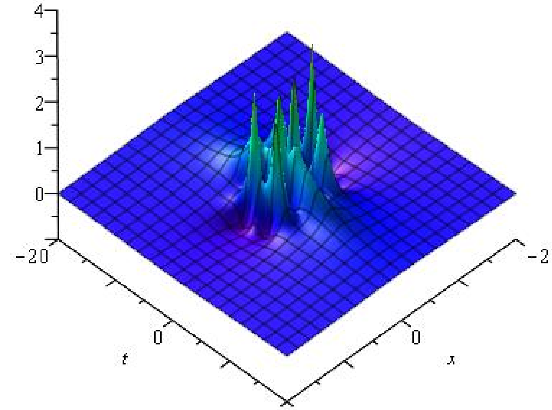
$$\widetilde{u}_4(x,t;\alpha,\beta)$$



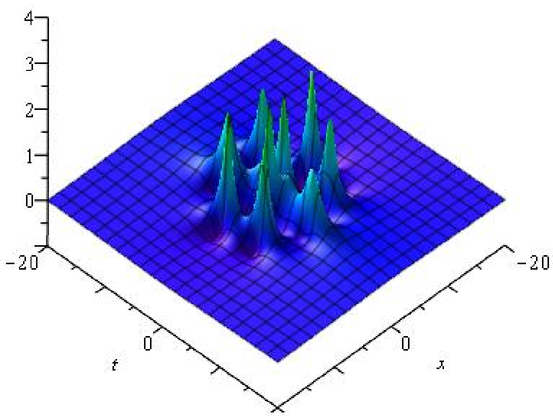
$\alpha = \beta = 0$



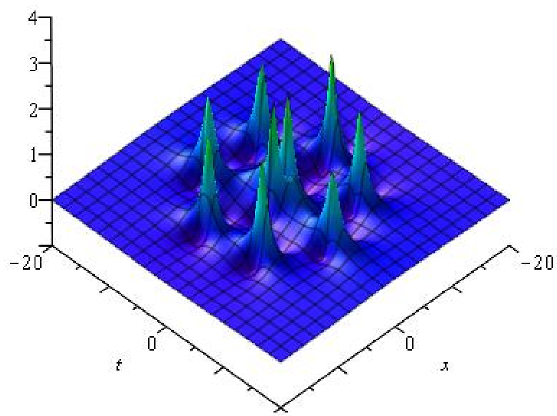
$\alpha = \beta = 10^3$



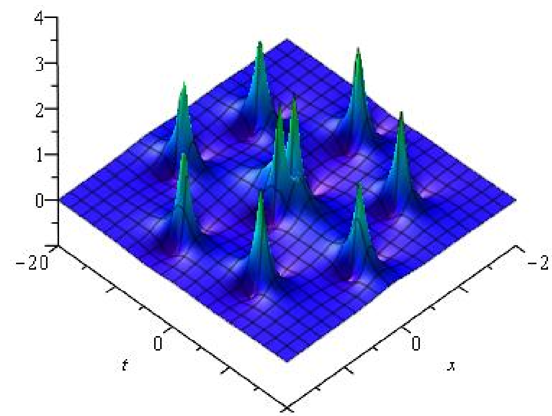
$\alpha = \beta = 10^5$



$\alpha = \beta = 10^6$



$\alpha = \beta = 10^7$



$\alpha = \beta = 10^8$

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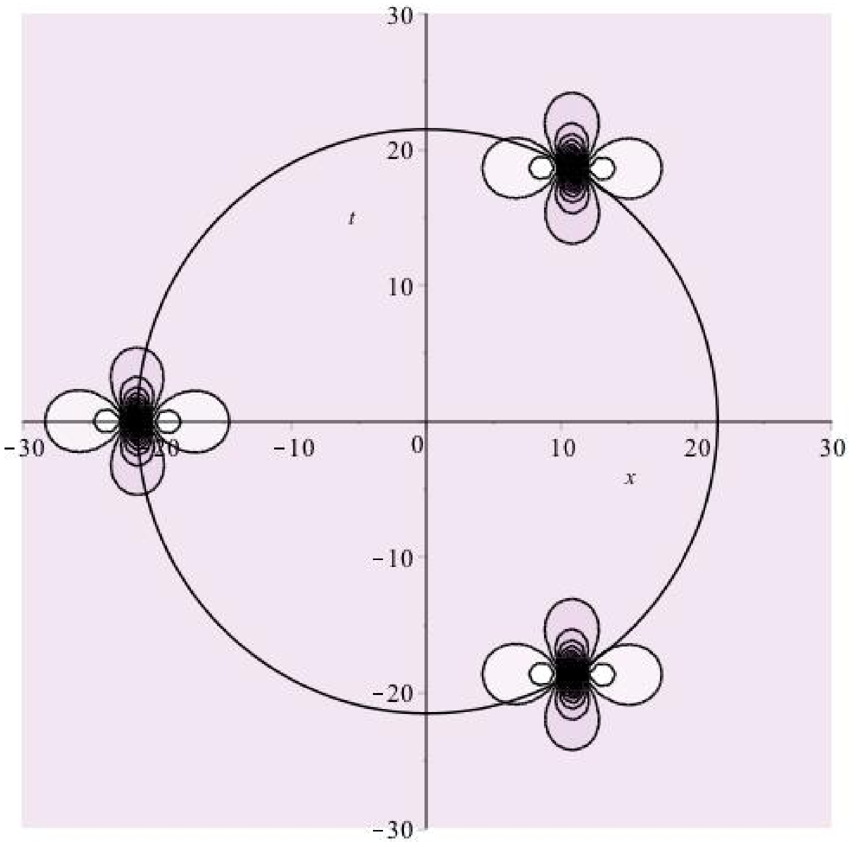
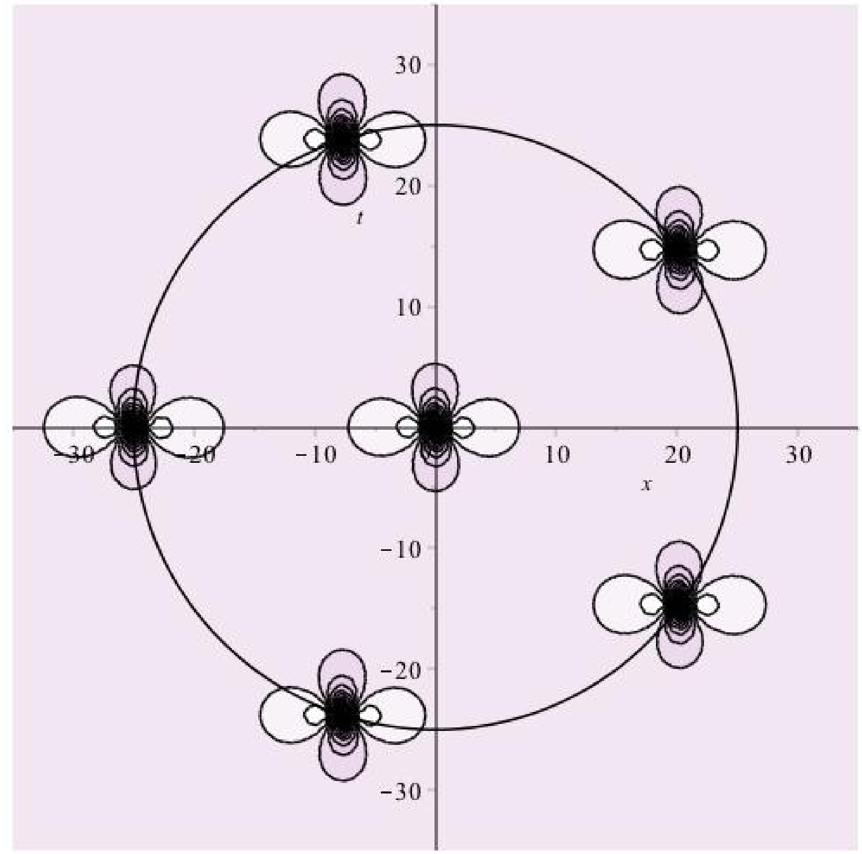


$\widetilde{u}_2(x,t;0,10^4)$



$\widetilde{u}_3(x,t;0,10^7)$

Theorem (**PAC & Dowie [2017]**)

The generalised rational solutions of the Boussinesq equation

$$u_{tt} + u_{xx} - (u^2)_{xx} - \tfrac{1}{3}u_{xxxx} = 0$$

have the form

$$\widetilde{u}_{n+1}(x,t;\alpha,\beta) = 2\frac{\partial^2}{\partial x^2}\ln \widetilde{f}_{n+1}(x,t;\alpha,\beta), \qquad n \geq 1$$

with

$$\widetilde{f}_{n+1}(x,t;\alpha,\beta) = f_{n+1}(x,t) + 2\alpha t p_n(x,t) + 2\beta x q_n(x,t) + (\alpha^2+\beta^2) f_{n-1}(x,t)$$

where $f_n(x,t)$, $p_n(x,t)$, $q_n(x,t)$ *are polynomials of degree* $n(n+1)$ *in* x *and* t.

Theorem (**PAC & Dowie [2017]**)

The generalised rational solutions of the Boussinesq equation

$$u_{tt} + u_{xx} - (u^2)_{xx} - \tfrac{1}{3}u_{xxxx} = 0$$

have the form

$$\widetilde{u}_{n+1}(x,t;\alpha,\beta) = 2\frac{\partial^2}{\partial x^2}\ln \widetilde{f}_{n+1}(x,t;\alpha,\beta), \qquad n \geq 1$$

with

$$\widetilde{f}_{n+1}(x,t;\alpha,\beta) = f_{n+1}(x,t) + 2\alpha t p_n(x,t) + 2\beta x q_n(x,t) + (\alpha^2+\beta^2) f_{n-1}(x,t)$$

where $f_n(x,t)$*,* $p_n(x,t)$*,* $q_n(x,t)$ *are polynomials of degree* $n(n+1)$ *in* x *and* t*.*

Theorem (**PAC & Dowie [2017]**)

The functions

$$\Theta_n^{\pm}(x,t) = tp_n(x,t) \pm \mathrm{i}xq_n(x,t), \qquad n \geq 1$$

are also solutions of the bilinear equation

$$\left(\mathrm{D}_t^2 + \mathrm{D}_x^2 - \tfrac{1}{3}\mathrm{D}_x^4\right) f \bullet f = 0$$

i.e. the ***same*** *bilinear equation as satisfied by* $f_n(x,t)$ *and* $\widetilde{f}_n(x,t;\alpha,\beta)$*.*

Nonlinear Superposition of Solutions

Corollary (**PAC & Dowie [2017]**)

The generalised rational solutions of the Boussinesq equation

$$u_{tt} + u_{xx} - (u^2)_{xx} - \tfrac{1}{3}u_{xxxx} = 0$$

have the form

$$\widetilde{u}_{n+1}(x,t;\alpha,\beta) = 2\frac{\partial^2}{\partial x^2}\ln \widetilde{f}_{n+1}(x,t;\alpha,\beta), \qquad n \geq 1$$

with

$$\widetilde{f}_{n+1}(x,t;\alpha,\beta) = f_{n+1}(x,t) + (\alpha + \mathrm{i}\beta)\Theta_n^+(x,t) + (\alpha - \mathrm{i}\beta)\Theta_n^-(x,t) + (\alpha^2+\beta^2)f_{n-1}(x,t)$$

where $\widetilde{f}_{n+1}(x,t;\alpha,\beta)$, $f_{n+1}(x,t)$, $\Theta_n^+(x,t)$, $\Theta_n^-(x,t)$ *and* $f_{n-1}(x,t)$ *are **all** independent solutions of the bilinear equation*

$$\left(\mathrm{D}_t^2 + \mathrm{D}_x^2 - \tfrac{1}{3}\mathrm{D}_x^4\right) f \bullet f = 0$$

Theorem (**Ankiewicz, Bassom, PAC & Dowie [2017]**)

Suppose that $u_n(x,t)$ *is a rogue wave solution of the Boussinesq equation*

$$u_{tt} + u_{xx} - (u^2)_{xx} - \tfrac{1}{3}u_{xxxx} = 0$$

then

$$\frac{1}{8\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_n^2(x,t)\,\mathrm{d}x\,\mathrm{d}t = \tfrac{1}{2}n(n+1)$$

and

$$\frac{1}{8\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_n^3(x,t)\,\mathrm{d}x\,\mathrm{d}t = n(n+1)$$

Theorem (**Ankiewicz, Bassom, PAC & Dowie [2017]**)

Suppose that $u_n(x,t)$ is a rogue wave solution of the Boussinesq equation

$$u_{tt} + u_{xx} - (u^2)_{xx} - \tfrac{1}{3}u_{xxxx} = 0$$

then

$$\frac{1}{8\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_n^2(x,t)\,\mathrm{d}x\,\mathrm{d}t = \tfrac{1}{2}n(n+1)$$

and

$$\frac{1}{8\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_n^3(x,t)\,\mathrm{d}x\,\mathrm{d}t = n(n+1)$$

Conjecture (**Ankiewicz & Akhmediev [2015]**)

Suppose that $\psi_n(x,t)$ is a rogue wave solution of the NLS equation

$$\mathrm{i}\psi_t + \psi_{xx} + \tfrac{1}{2}|\psi|^2\psi = 0$$

then

$$\frac{1}{8\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[|\psi_n^2(x,t)| - 1\right]^2 \mathrm{d}x\,\mathrm{d}t = \tfrac{1}{2}n(n+1)$$

$$u_n(x,t) = 2\frac{\partial^2}{\partial x^2}\ln f_n^{\mathrm{bq}}(x,t), \qquad |\psi_n^2(x,t)| - 1 = 2\frac{\partial^2}{\partial x^2}\ln F_n^{\mathrm{nls}}(x,t)$$

Conservation Laws

Definition. A **conservation law** is an equation of the form

$$\frac{\partial T}{\partial t} + \frac{\partial X}{\partial x} = 0$$

where $T(x,t)$ is the conserved density and $X(x,t)$ the associated flux. The integral

$$\int_{-\infty}^{\infty} T(x,t)\,\mathrm{d}x = c$$

with c a constant, is called a constant of motion, with t interpreted as a timelike variable. It follows that

$$\int_{-\infty}^{\infty} X(x,t)\,\mathrm{d}t = k$$

with k also a constant.

To study conservation laws for the Boussinesq equation, we consider the system

$$\begin{aligned} &u_t + v_x = 0 \\ &v_t + (u^2)_x - u_x + \tfrac{1}{3}u_{xxx} = 0 \end{aligned}$$

The first few conserved densities $T_j(x,t)$ and associated fluxes $X_j(x,t)$ for the system are

$$\begin{aligned}
T_1(x,t) &= u, & X_1(x,t) &= v \\
T_2(x,t) &= v, & X_2(x,t) &= u^2 - u + \tfrac{1}{3}u_{xx} \\
T_3(x,t) &= uv, & X_3(x,t) &= \tfrac{2}{3}u^3 + \tfrac{1}{2}v^2 - \tfrac{1}{2}u^2 - \tfrac{1}{6}u_x^2 + \tfrac{1}{3}uu_{xx} \\
T_4(x,t) &= \tfrac{2}{3}u^3 + v^2 - u^2 - \tfrac{1}{3}u_x^2, & X_4(x,t) &= 2u^2v - 2uv + \tfrac{2}{3}vu_{xx} - \tfrac{2}{3}u_xv_x
\end{aligned}$$

Hence the first few constants of the motion are

$$\int_{-\infty}^{\infty} u(x,t)\,\mathrm{d}x = c_1, \qquad \int_{-\infty}^{\infty} u(x,t)v(x,t)\,\mathrm{d}x = c_3$$

$$\int_{-\infty}^{\infty} v(x,t)\,\mathrm{d}x = c_2, \qquad \int_{-\infty}^{\infty} \left(\tfrac{2}{3}u^3 + v^2 - u^2 - \tfrac{1}{3}u_x^2\right)\mathrm{d}x = c_4$$

with c_1, c_2, c_3 and c_4 constants, and the associated fluxes are

$$\int_{-\infty}^{\infty} v(x,t)\,\mathrm{d}t = k_1, \qquad \int_{-\infty}^{\infty} \left(\tfrac{2}{3}u^3 + \tfrac{1}{2}v^2 - \tfrac{1}{2}u^2 - \tfrac{1}{6}u_x^2 + \tfrac{1}{3}uu_{xx}\right)\mathrm{d}t = k_3$$

$$\int_{-\infty}^{\infty} \left(u^2 - u + \tfrac{1}{3}u_{xx}\right)\mathrm{d}t = k_2, \qquad \int_{-\infty}^{\infty} \left(2u^2v - 2uv + \tfrac{2}{3}vu_{xx} - \tfrac{2}{3}u_xv_x\right)\mathrm{d}t = k_4$$

with k_1, k_2, k_3 and k_4 constants. For the algebraically decaying rational solutions of the Boussinesq equation then $c_j = 0$ and $k_j = 0$, for $j = 1, \ldots, 4$.

Rational Solutions of the Kadomstev-Petviashvili I Equation

$$(V_\tau + 6VV_\xi + V_{\xi\xi\xi})_\xi - 3V_{\eta\eta} = 0$$

Kadomstev-Petviashvili Equation

$$(V_\tau + 6VV_\xi + V_{\xi\xi\xi})_\xi + 3\sigma^2 V_{\eta\eta} = 0, \qquad \sigma^2 = \pm 1$$

- The first $2+1$-dimensional equation found to be solvable by inverse scattering (**Dryuma [1974], Zakharov & Shabat [1974]**).

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- The case $\sigma = \mathrm{i}$ is known as the **KPI equation** and the case $\sigma = 1$ is known as the **KPII equation**. Inverse scattering is different for the two cases:
  - Riemann-Hilbert method for KPI (**Manakov [1981], Segur [1982], Fokas & Ablowitz [1983]**),
  - $\bar{\partial}$ method for KPII (**Ablowitz, Bar Yaacov & Fokas [1983]**).

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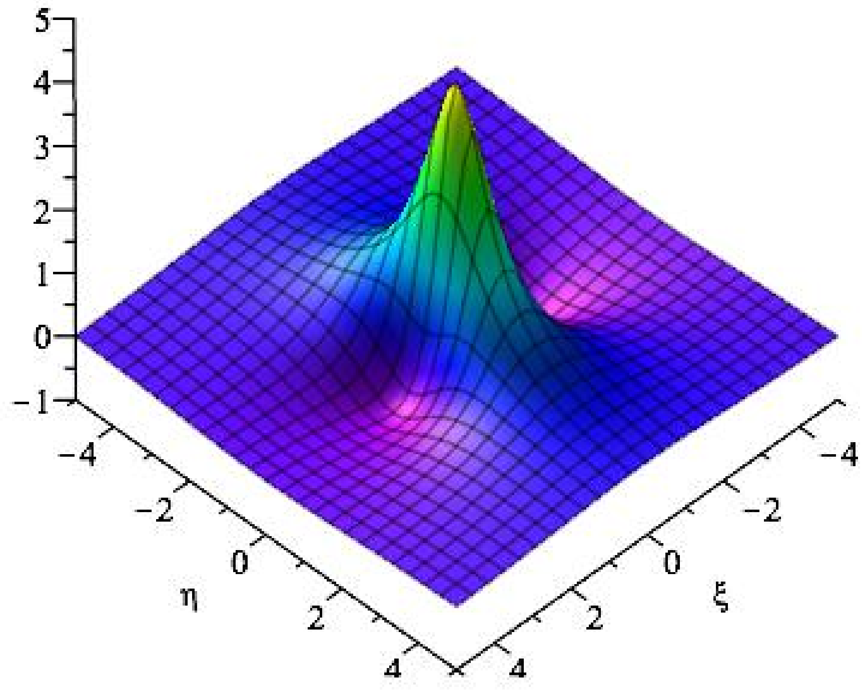
  - $\bar{\partial}$ method for KPII (**Ablowitz, Bar Yaacov & Fokas [1983]**).
- Arises in several physical applications:
  - Derived by **Kadomtsev & Petviashvili [1970]** to model ion-acoustic waves of small amplitude propagating in plasmas.
  - Surface water waves (**Ablowitz & Segur [1979]**).
  - Two-dimensional shallow water waves (**Segur & Finkel [1985], Hammack *et al.* [1989]**).

It is well-known that KPI

$$(V_\tau + 6VV_\xi + V_{\xi\xi\xi})_\xi - 3V_{\eta\eta} = 0$$

has the 1-lump solution (**Manakov *et al.* [1977]**)

$$V(\xi,\eta,\tau) = 2\frac{\partial^2}{\partial\xi^2}\ln\{(\xi-3\tau)^2+\eta^2+1\} = -4\frac{(\xi-3\tau)^2-\eta^2-1}{\{(\xi-3\tau)^2+\eta^2+1\}^2}$$



The focusing NLS equation

$$\mathrm{i}\psi_t + \psi_{xx} + \tfrac{1}{2}|\psi|^2\psi = 0 \tag{1}$$

has the rational solutions in the form

$$\psi(x,t;\alpha,\beta) = \left\{1 - 4\frac{G(x,t;\alpha,\beta) + \mathrm{i}H(x,t;\alpha,\beta)}{F(x,t;\alpha,\beta)}\right\}\exp\left(\tfrac{1}{2}\mathrm{i}t\right)$$

therefore

$$\begin{aligned}|\psi(x,t;\alpha,\beta)|^2 - 1 &= \frac{16G^2(x,t;\alpha,\beta) + 16H^2(x,t;\alpha,\beta) - 8F(x,t;\alpha,\beta)G(x,t;\alpha,\beta)}{F^2(x,t;\alpha,\beta)} \\ &= 4\frac{\partial^2}{\partial x^2}\ln F(x,t;\alpha,\beta)\end{aligned}$$

Dubard & Matveev [2011, 2013] (see also **Gaillard [2016]**) show that

$$\begin{aligned}V(\xi,\eta,\tau) &= 2\frac{\partial^2}{\partial \xi^2}\ln F(\xi-3\tau,\eta;\alpha,-48\tau) \\ &= \tfrac{1}{2}\left(|\psi(x,t;\alpha,\beta)|^2 - 1\right)\Big|_{x=\xi-3\tau,t=\eta,\beta=-48\tau}\end{aligned}$$

is a solution of the KPI equation

$$(V_\tau + 6VV_\xi + V_{\xi\xi\xi})_\xi - 3V_{\eta\eta} = 0 \tag{2}$$

This relates solutions of the focusing NLS equation (1) and KPI (2).

Let $x = \xi - 3\tau$, $t = \eta$ and $\beta = -48\tau$ in

$$F_2^{\text{nls}}(x, t; \alpha, \beta) = x^6 + 3(t^2+1)x^4 + 3(t^2-3)^2x^2 + t^6 + 27t^4 + 99t^2 + 9 \\ + 2\alpha t(3x^2 - t^2 - 9) + \alpha^2 - 2\beta x(x^2 - 3t^2 - 3) + \beta^2$$

then

$$F_2(\xi, \eta, \tau; \alpha) = F_2^{\text{nls}}(\xi - 3\tau, \eta; \alpha, -48\tau)$$

satisfies

$$\left(\mathrm{D}_\xi^4 + \mathrm{D}_\xi \mathrm{D}_\tau - 3\mathrm{D}_\eta^2\right) F_2 \bullet F_2 = 0$$

which is the bilinear form of the KPI equation

$$(V_\tau + 6VV_\xi + V_{\xi\xi\xi})_\xi - 3V_{\eta\eta} = 0 \qquad (2)$$

Therefore

$$V(\xi, \eta, \tau; \alpha) = 2\frac{\partial^2}{\partial \xi^2} \ln F_2(\xi, \eta, \tau; \alpha)$$

is a rational solution of the KPI equation (2).

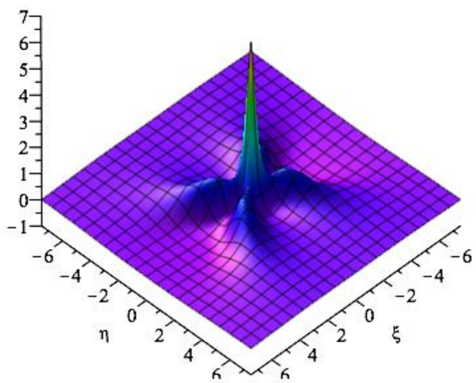
Further

$$\int_{-\infty}^{\infty} V(\xi, \eta, \tau)\, \mathrm{d}\xi = 0$$

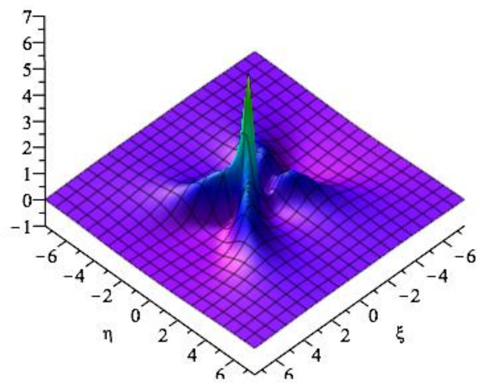
and

$$V(\xi, \eta, \tau) \to 0, \qquad \text{as} \qquad \xi^2 + \eta^2 \to \infty$$

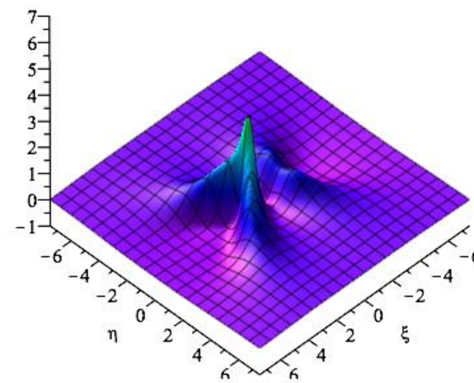
Rational Solutions of KPI



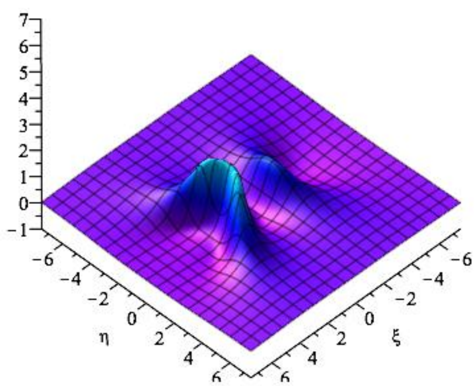
$\tau = 0$



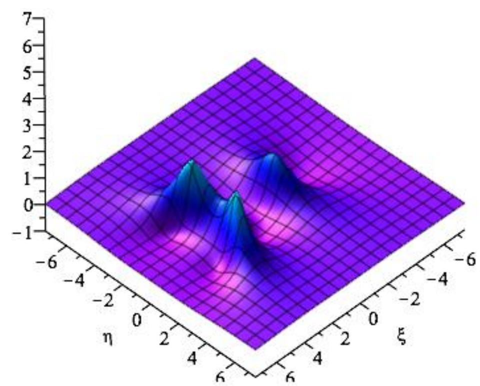
$\tau = 0.05$



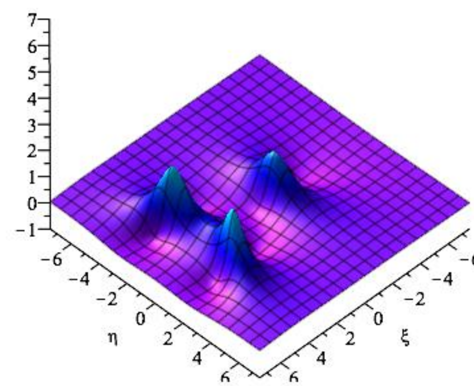
$\tau = 0.1$



$\tau = 0.2$



$\tau = 0.4$



$\tau = 0.7$

Reductions of KPI to the Boussinesq equation

If in KPI

$$(V_\tau + 6VV_\xi + V_{\xi\xi\xi})_\xi - 3V_{\eta\eta} = 0$$

we make the reduction

$$V(\xi, \eta, \tau) = u(x, t), \qquad x = \xi - 3\tau, \quad t = \eta$$

then we obtain the Boussinesq equation

$$u_{tt} + u_{xx} - (u^2)_{xx} - \tfrac{1}{3}u_{xxxx} = 0$$

Hence, if

$$u(x, t) = 2\frac{\partial^2}{\partial x^2} \ln f^{\mathrm{bq}}(x, t)$$

is a solution of the Boussinesq equation, then

$$V(\xi, \eta, \tau) = 2\frac{\partial^2}{\partial \xi^2} \ln f^{\mathrm{bq}}(\xi - 3\tau, \eta)$$

is a solution of KPI.

Reductions of KPI to the Boussinesq equation

If in KPI

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then we obtain the Boussinesq equation

$$u_{tt} + u_{xx} - (u^2)_{xx} - \tfrac{1}{3}u_{xxxx} = 0$$

Hence, if

$$u(x,t) = 2\frac{\partial^2}{\partial x^2}\ln f^{\mathrm{bq}}(x,t)$$

is a solution of the Boussinesq equation, then

$$V(\xi,\eta,\tau) = 2\frac{\partial^2}{\partial \xi^2}\ln f^{\mathrm{bq}}(\xi - 3\tau,\eta)$$

is a solution of KPI.

- If $f_1^{\mathrm{bq}}(x,t) = x^2 + t^2 + 1$ then we obtain the 1-lump solution of KPI

$$V(\xi,\eta,\tau) = 2\frac{\partial^2}{\partial \xi^2}\ln\{(\xi-3\tau)^2 + \eta^2 + 1\} = -4\frac{(\xi-3\tau)^2 - \eta^2 - 1}{\{(\xi-3\tau)^2 + \eta^2 + 1\}^2}$$

Using the second rational solution of the Boussinesq equation we obtain the KPI rational solution

$$V(\xi,\eta,\tau;\alpha,\beta) = 2\frac{\partial^2}{\partial\xi^2}\ln f_2^{\text{bq}}(\xi,\eta,\tau;\alpha,\beta)$$

where

$$\begin{aligned}
f_2^{\text{bq}}(\xi,\eta,\tau;\alpha,\beta) = \xi^6 &- 18\tau\xi^5 + 3\left(45\tau^2+\eta^2+\tfrac{25}{9}\right)\xi^4 - 12\left(45\tau^2+3\eta^2+\tfrac{25}{3}\right)\tau\xi^3 \\
&+ \left\{3\eta^4 + 18\left(9\tau^2+\tfrac{5}{3}\right)\eta^2 + 1215\tau^4 + 450\tau^2 - \tfrac{125}{9}\right\}\xi^2 \\
&- \left\{18\tau\eta^4 + 36\left(9\tau^2+5\right)\tau\eta^2 + 1458\tau^5 + 900\tau^3 + \tfrac{250}{3}\tau\right\}\xi \\
&+ \eta^6 + 27\left(\tau^2+\tfrac{17}{81}\right)\eta^4 + 9\left(27\tau^4+30\tau^2+\tfrac{475}{81}\right)\eta^2 \\
&+ 729\tau^6 + 675\tau^4 - 125\tau^2 + \tfrac{625}{9} \\
&+ 2\alpha\left\{3\xi^2\eta - 18\xi\tau\eta - \eta^3 + \left(27\tau^2+\tfrac{5}{3}\right)\eta\right\} \\
&+ 2\beta\left\{\xi^3 - 9\xi^2\tau - \left(3\eta^2-27\tau^2+\tfrac{1}{3}\right)\xi - 27\tau^3 + 9\tau\eta^2 + \tau\right\} \\
&+ \alpha^2 + \beta^2
\end{aligned}$$

Compare $F_2^{\mathrm{nls}}(\xi,\eta,\tau;\alpha)$ and $f_2^{\mathrm{bq}}(\xi,\eta,\tau;\alpha,\beta)$

$$\begin{aligned}F_2^{\mathrm{nls}}(\xi,\eta,\tau;\alpha) = \xi^6 &- 18\tau\xi^5 + 3\left(45\tau^2+\eta^2+1\right)\xi^4 - 12\left(45\tau^2+3\eta^2-5\right)\tau\xi^3\\
&+\left\{3\eta^4+18\left(9\tau^2-1\right)\eta^2+1215\tau^4-702\tau^2+27\right\}\xi^2\\
&-\left\{18\tau\eta^4+36\left(9\tau^2+5\right)\tau\eta^2+1458\tau^5-2268\tau^3+450\tau\right\}\xi\\
&+\eta^6+27\left(\tau^2+1\right)\eta^4+9\left(27\tau^4+78\tau^2+11\right)\eta^2\\
&+729\tau^6-2349\tau^4+3411\tau^2+9\\
&+2\alpha\left\{3\xi^2\eta-18\xi\tau\eta-\eta^3+9\left(3\tau^2-1\right)\eta\right\}+\alpha^2\end{aligned}$$

$$\begin{aligned}f_2^{\mathrm{bq}}(\xi,\eta,\tau;\alpha,\beta) = \xi^6 &- 18\tau\xi^5 + 3\left(45\tau^2+\eta^2+\tfrac{25}{9}\right)\xi^4 - 12\left(45\tau^2+3\eta^2+\tfrac{25}{3}\right)\tau\xi^3\\
&+\left\{3\eta^4+18\left(9\tau^2+\tfrac{5}{3}\right)\eta^2+1215\tau^4+450\tau^2-\tfrac{125}{9}\right\}\xi^2\\
&-\left\{18\tau\eta^4+36\left(9\tau^2+5\right)\tau\eta^2+1458\tau^5+900\tau^3+\tfrac{250}{3}\tau\right\}\xi\\
&+\eta^6+27\left(\tau^2+\tfrac{17}{81}\right)\eta^4+9\left(27\tau^4+30\tau^2+\tfrac{475}{81}\right)\eta^2\\
&+729\tau^6+675\tau^4-125\tau^2+\tfrac{625}{9}\\
&+2\alpha\left\{3\xi^2\eta-18\xi\tau\eta-\eta^3+9\left(3\tau^2+\tfrac{5}{9}\right)\eta\right\}\\
&+2\beta\left\{\xi^3-9\xi^2\tau-\left(3\eta^2-27\tau^2+\tfrac{1}{3}\right)\xi-27\tau^3+9\tau\eta^2+\tau\right\}\\
&+\alpha^2+\boldsymbol{\beta^2}\end{aligned}$$

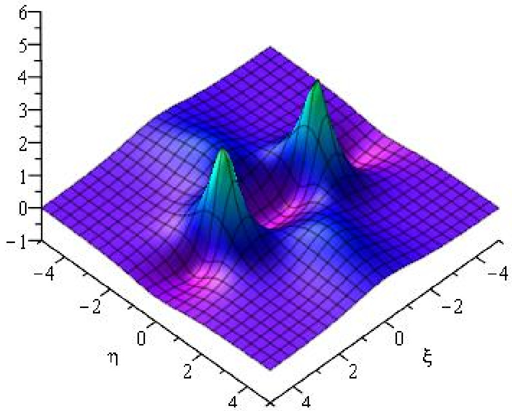
Now consider the general expression, with parameters μ, α and β

$$
\begin{aligned}
F_2^{\rm gen}(\xi,\eta,\tau;\mu,\alpha,\beta) = \xi^6 &- 18\tau\xi^5 + (3\eta^2+135\tau^2-6\mu^2+9)\xi^4 \\
&- \left\{36\tau\eta^2+540\tau^3-12(6\mu^2+6\mu-7)\tau\right\}\xi^3 \\
&+ \left\{3\eta^4+18(9\tau^2-2\mu+1)\eta^2+1215\tau^4\right. \\
&\qquad \left.-54(6\mu^2+12\mu-5)\tau^2+9\mu(\mu+2)(\mu^2-2\mu+2)\right\}\xi^2 \\
&- \left\{18\tau\eta^4+36(9\tau^2+5)\tau\eta^2+1458\tau^5-324(2\mu^2+6\mu-1)\tau^3\right. \\
&\qquad \left.+18\mu(3\mu^3+12\mu^2-2\mu+12)\tau\right\}\xi \\
&+ \eta^6+(27\tau^2+6\mu^2+12\mu+9)\eta^4 \\
&+ \left\{243\tau^4+54(6\mu+7)\tau^2+9(\mu^4+4\mu^3+6\mu^2-4\mu+4)\right\}\eta^2 \\
&+ 729\tau^6-81(\mu^2+24\mu-1)\tau^4 \\
&+ 9(9\mu^4+72\mu^3+150\mu^2+132\mu+16)\tau^2+9(\mu^2-2\mu+2)^2 \\
&+ 2\alpha\left\{3\eta\xi^2-18\tau\eta\xi-\eta^3+3\left[9\tau^2-\mu(\mu+2)\right]\eta\right\} \\
&+ 2\beta\left\{\xi^3-9\tau\xi^2-6(\eta^2-9\tau^2+\mu^2)\xi+9\tau\eta^2-27\tau^3\right. \\
&\qquad \left.+3(3\mu^2+12\mu+4)\tau\right\}+\alpha^2+\beta^2
\end{aligned}
$$

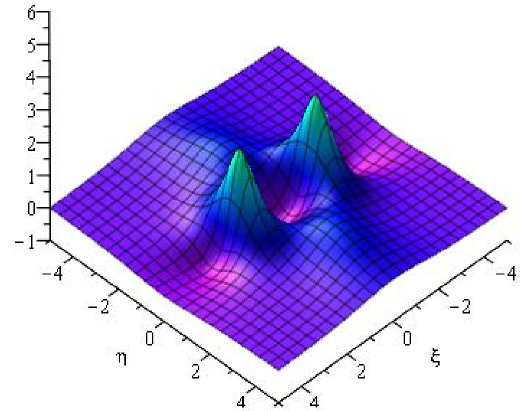
which has both $F_2^{\rm nls}(\xi,\eta,\tau;\alpha)$ and $f_2^{\rm bq}(\xi,\eta,\tau;\alpha,\beta)$ as special cases:

$$
\begin{aligned}
F_2^{\rm nls}(\xi,\eta,\tau;\alpha) &= F_2^{\rm gen}(\xi,\eta,\tau;1,\alpha,0) \\
f_2^{\rm bq}(\xi,\eta,\tau;\alpha,\beta) &= F_2^{\rm gen}(\xi,\eta,\tau;-\tfrac{1}{3},\alpha,\beta)
\end{aligned}
$$

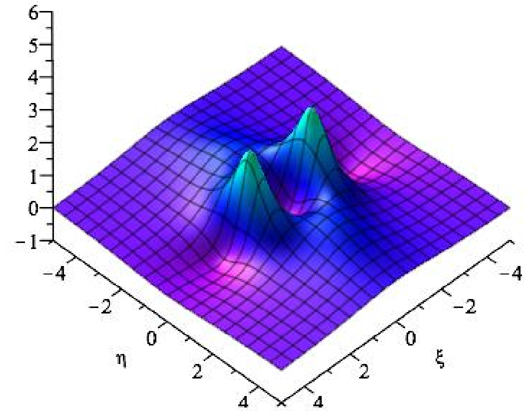
$$v_2(\xi, \eta, 0; \mu, 0, 0)$$



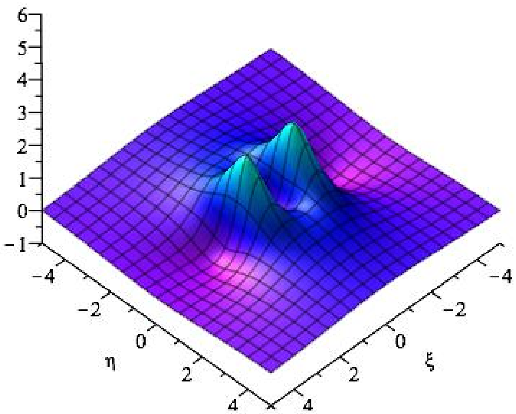
$\mu = -1$



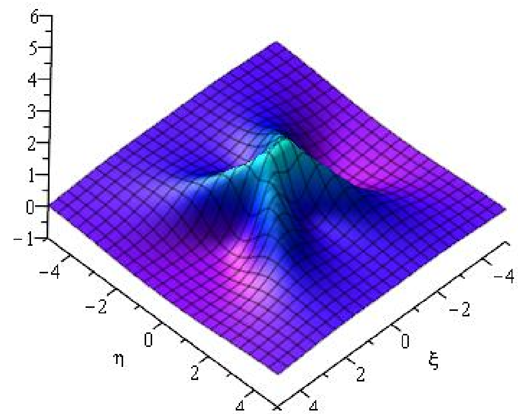
$\mu = -2/3$



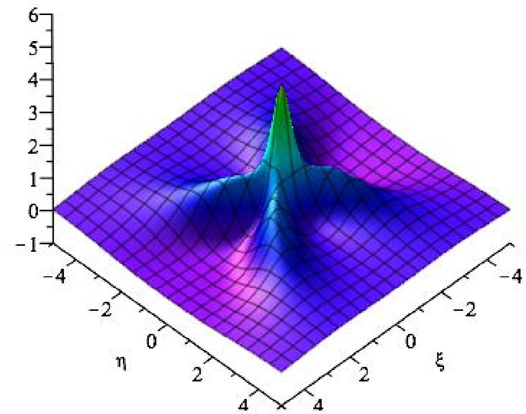
$\mu = -1/3$



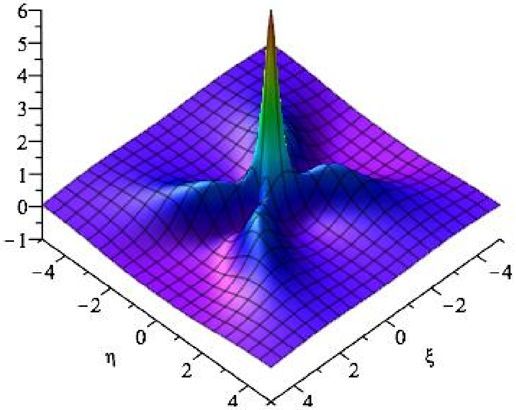
$\mu = 0$



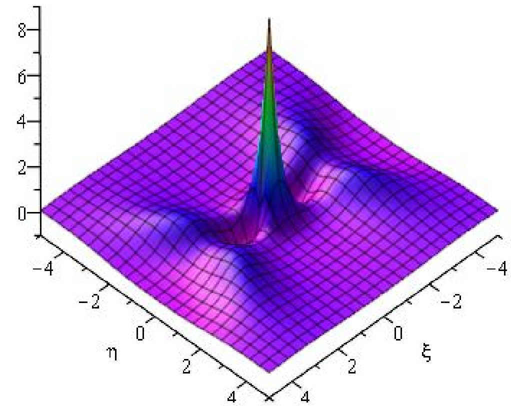
$\mu = 0.5115960325$



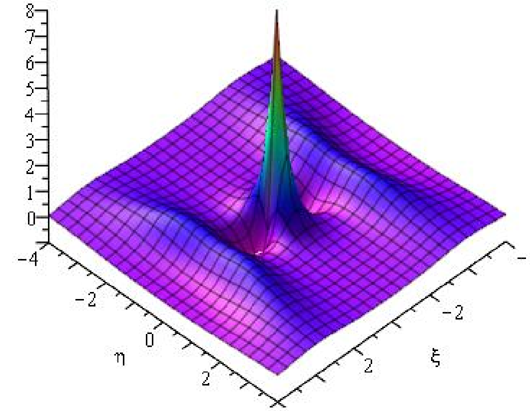
$\mu = 3/4$



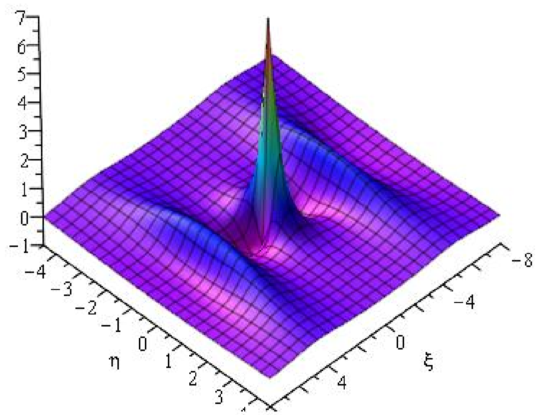
$\mu = 1$



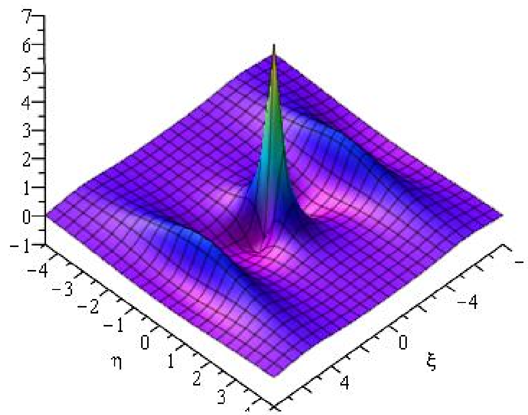
$\mu = \frac{1}{2}(1+\sqrt{5})$



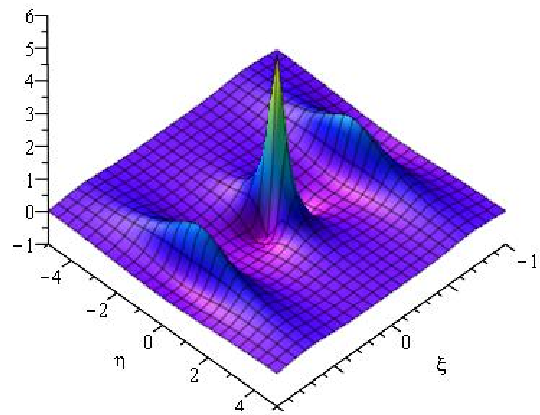
$\mu = 2$



$\mu = 2.5$



$\mu = 3$



$\mu = 4$

For $\mu < \mu^*$, the solution $v_2(\xi, \eta, 0; \mu, 0, 0)$ has two peaks on the line $\eta = 0$, which coalesce when $\mu = \mu^*$ to form one peak at $\xi = \eta = 0$. By considering when

$$\frac{\partial^2}{\partial \xi^2} V(\xi, 0, 0; \mu, 0, 0)\bigg|_{\xi=0} = -\frac{8(3\mu^4 + 12\mu^3 + 16\mu^2 - 6)}{(\mu^2 - 2\mu + 2)^2} = 0$$

then μ^* is the real positive root of

$$\begin{aligned} &3\mu^4 + 12\mu^3 + 16\mu^2 - 6 \\ &\quad = 3\left[\mu^2 + 2(1 - \tfrac{1}{3}\sqrt{6})\mu + 2 - \sqrt{6}\right]\left[\mu^2 + 2(1 + \tfrac{1}{3}\sqrt{6})\mu + 2 + \sqrt{6}\right] = 0 \end{aligned}$$

i.e.

$$\mu^* = -1 + \tfrac{1}{3}\sqrt{6} + \tfrac{1}{3}\sqrt{-3 + 3\sqrt{6}} \approx 0.5115960325$$

For $\mu > \mu^*$, it can be shown that

$$V(0, 0, 0; \mu, 0, 0) = \frac{4\mu(\mu + 2)}{\mu^2 - 2\mu + 2}$$

increases until it reaches a maximum height of $4(2 + \sqrt{5})$ when $\mu = \frac{1}{2}(1 + \sqrt{5})$, which is the golden mean!

Ablowitz, Chakravarty, Trubatch & Villaroel [2000] show that KPI

$$(V_\tau + 6VV_\xi + V_{\xi\xi\xi})_\xi - 3V_{\eta\eta} = 0$$

has rational solutions in the form

$$V_m(\xi,\eta,\tau) = 2\frac{\partial^2}{\partial\xi^2}\ln F_m(\xi,\eta,\tau)$$

where $F_m(\xi,\eta,\tau)$ is a polynomial of degree $2m$ in ξ, η and τ given by

$$F_m(\xi,\eta,\tau) = 4^n\sum_{j=0}^{2n}\frac{\partial^j}{\partial\xi^j}|p_m(\xi,\eta,\tau)|^2$$

with $p_m(\xi,\eta,\tau)$ polynomials given by

$$p_m(\xi,\eta,\tau) = \exp\left\{-\tfrac{1}{2}\mathrm{i}(k\xi - \tfrac{1}{2}k^2\eta + k^3\tau)\right\}\frac{\mathrm{d}^m}{\mathrm{d}k^m}\exp\left\{\tfrac{1}{2}\mathrm{i}(k\xi - \tfrac{1}{2}k^2\eta + k^3\tau)\right\}\bigg|_{k=\mathrm{i}}$$

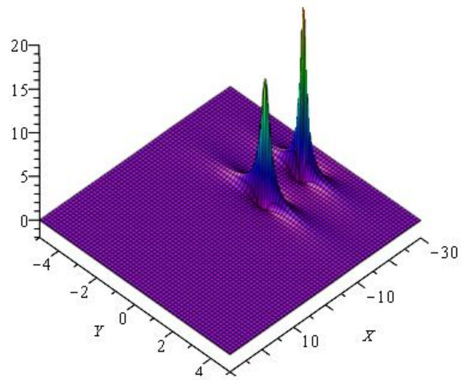
Hence

$$F_1(\xi,\eta,\tau) = (\xi - 3\tau + 1)^2 + \eta^2 + 1$$

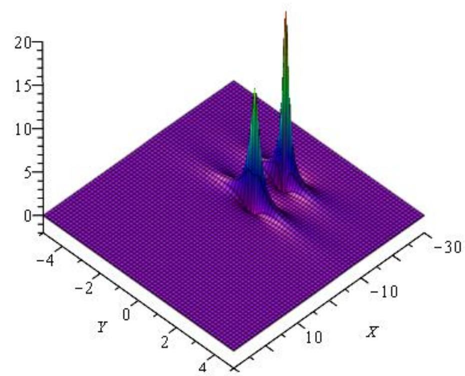
$$\begin{aligned}F_2(\xi,\eta,\tau) &= (\xi - 3\tau + 1)^4 + 2(\eta^2 + 12\tau + 6)\xi^2 - 4(3\tau + 1)(\eta^2 + 12\tau - 5)\xi \\ &\quad + \eta^4 + 6(3\tau - 2)\tau\eta^2 + 216\tau^3 + 54\tau^2 - 12\tau + 23\end{aligned}$$

Rational Solutions of KPI

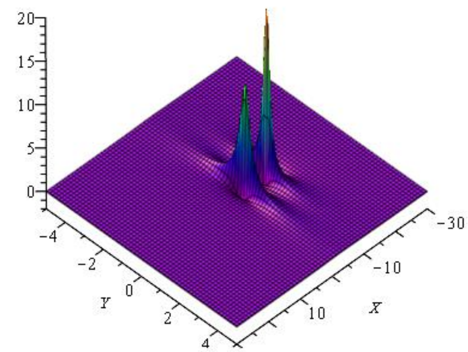
$V_2(\xi, \eta, \tau)$



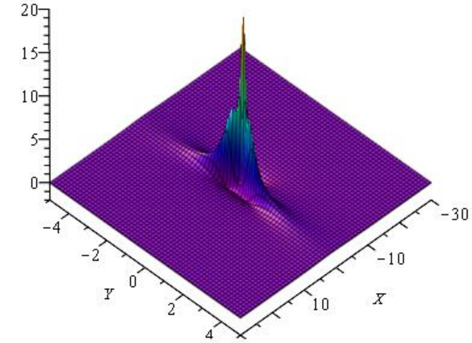
$\tau = -1.5$



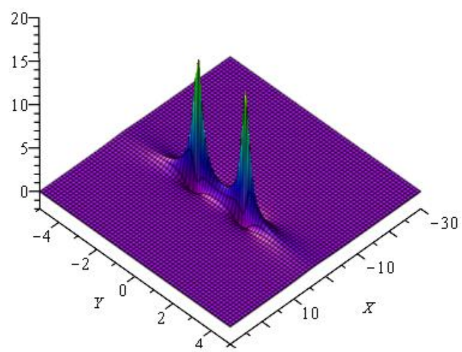
$\tau = 1$



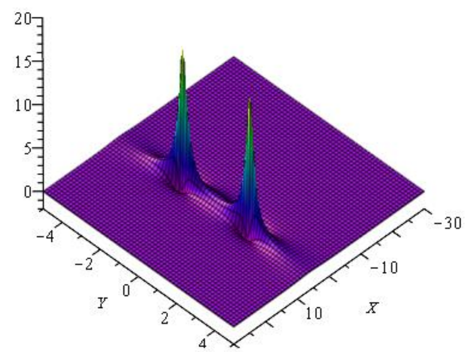
$\tau = -0.5$



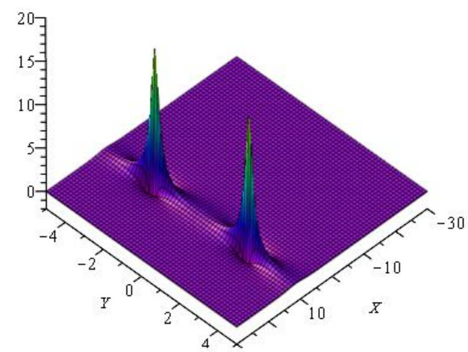
$\tau = 0$



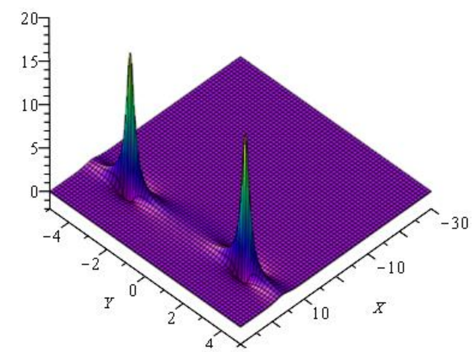
$\tau = 0.25$



$\tau = 0.5$

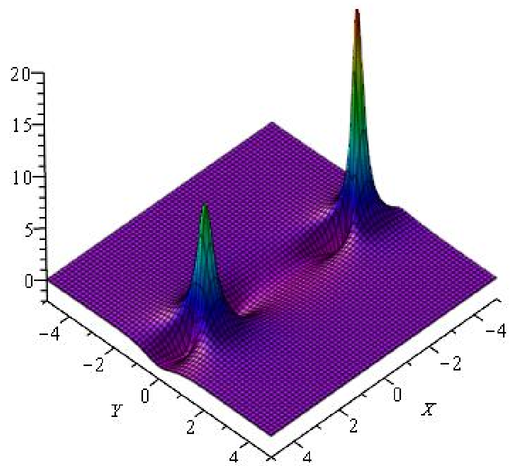


$\tau = 1$

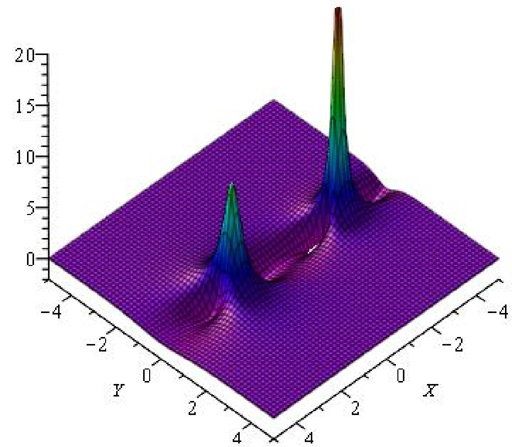


$\tau = 1.5$

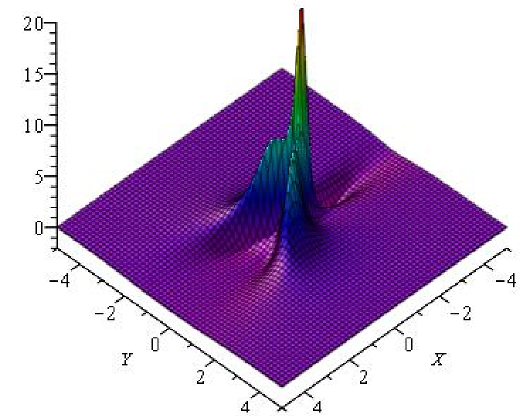
$$V_2(\xi - 3\tau, \eta, \tau)$$



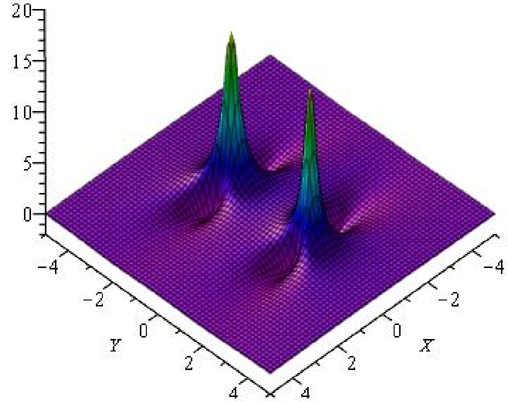
$\tau = -0.5$



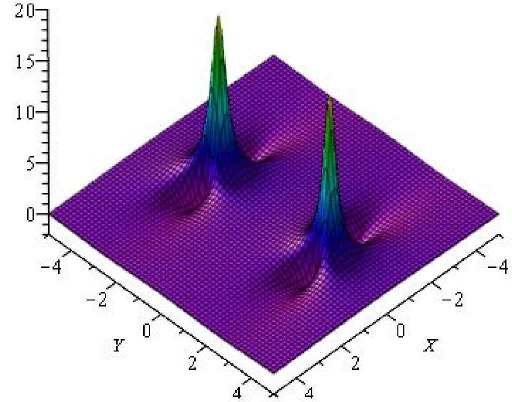
$\tau = -0.25$



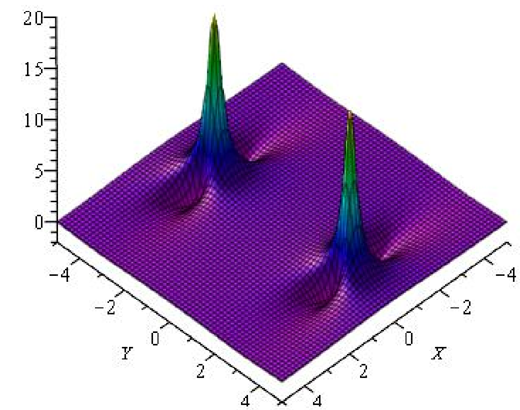
$\tau = 0$



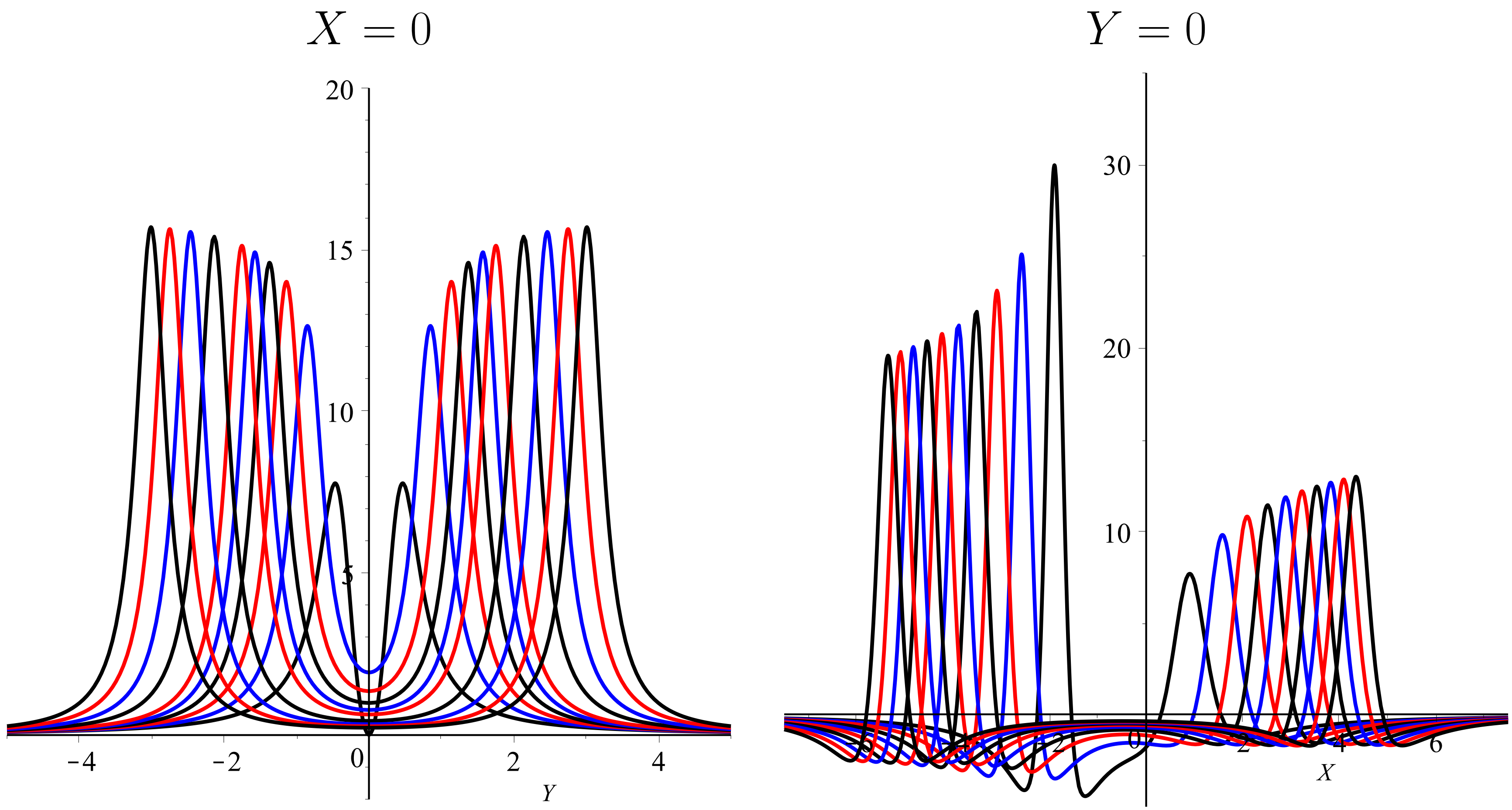
$\tau = 0.5$



$\tau = 1$



$\tau = 1.5$



$$\lim_{\tau\to\infty}\max_{Y\in\mathbb{R}}[V_2(0,Y,\tau)] = 16 = \lim_{\tau\to-\infty}\max_{X\in\mathbb{R}}[V_2(X,0,\tau)]$$

The rational solutions of KPI

$$(V_\tau + 6VV_\xi + V_{\xi\xi\xi})_\xi - 3V_{\eta\eta} = 0 \tag{1}$$

obtained by **Ablowitz, Chakravarty, Trubatch & Villaroel [2000]** are derived in terms of the eigenfunctions of the non-stationary Schrödinger equation

$$\mathrm{i}\varphi_\eta + \varphi_{\xi\xi} + V\varphi = 0 \tag{2}$$

with potential $V = V(\xi, \eta, \tau)$, which is used in the solution of KPI by inverse scattering. KPI (1) is obtained from the compatibility of (2) and

$$\varphi_\tau + 4\varphi_{\xi\xi\xi} + 6V\varphi_\xi + W\varphi = 0, \qquad W_\xi = V \tag{3}$$

These rational solutions of KPI are deeply connected with an integer called the "charge" or "index", and this number is related to the degree of the polynomial that generates the rational solution.

Conjecture

Suppose that $V_m(\xi, \eta, \tau)$ is a rational solution of the KPI equation (1) derived in terms of the eigenfunctions of the non-stationary Schrödinger equation (2), then

$$\frac{1}{8\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} V_m^2(\xi, \eta, \tau)\,\mathrm{d}\xi\,\mathrm{d}\eta = m$$

Numerical Results

m	$\frac{1}{8\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} V_m^2(\xi,\eta,\tau)\,\mathrm{d}\xi\,\mathrm{d}\eta$	$\frac{1}{8\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} V_m^3(\xi,\eta,\tau)\,\mathrm{d}\xi\,\mathrm{d}\eta$
1	1	2
2	2	4.15423119
3	3	6.87299527
4	4	9.88225790
5	5	13.07265607
6	6	16.38558786

Conjecture

Suppose that $V_m(\xi,\eta,\tau)$ is a rational solution of the KPI equation derived in terms of the eigenfunctions of the non-stationary Schrödinger equation, then

$$\frac{1}{8\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} V_m^2(\xi,\eta,\tau)\,\mathrm{d}\xi\,\mathrm{d}\eta = m$$

Conclusions

- There are algebraically decaying rational solutions of the **focusing NLS equation**, the **Boussinesq equation** and the **Kadomtsev-Petviashvili I equation** which appear to have applications in rogue or freak waves.
- The rational solutions of KPI have been derived using several methods:
  - from the NLS equation;
  - from the Boussinesq equation; and
  - from eigenfunctions of the associated spectral problem.

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- The rational solutions of KPI have been derived using several methods:
  - from the NLS equation;
  - from the Boussinesq equation; and
  - from eigenfunctions of the associated spectral problem.

Open Problems

- Can the polynomials associated with the rational solutions of the Boussinesq equation be expressed as determinants, or Wronskians?
- Are these rational solutions of the Boussinesq and KPI equations stable?
- Can the hierarchy of rational solutions of the Boussinesq equation be derived from its Lax pairs?
- Do these special polynomials associated with rational solutions of soliton equations have further applications, e.g. in numerical analysis?