# Localisation and delocalisation in the parabolic Anderson model

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University College London

joint works (2006 – 2017) with
Wolfgang König (TU Berlin and WIAS), Hubert Lacoin (IMPA), Peter Mörters (Bath), Stephen Muirhead (Kings), Marcel Ortgiese (Bath), Richard Pymar (Birkbeck), Aleksander Twarowski (London)

The Leslie Comri Seminar Series, University of Greenwich

4 October 2017

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How does $u(t,\cdot)$ behave as $t \to \infty$?

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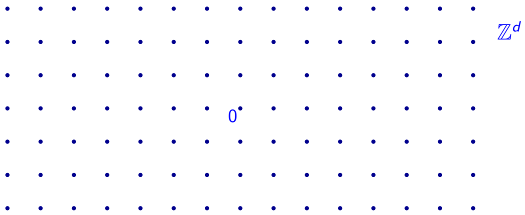

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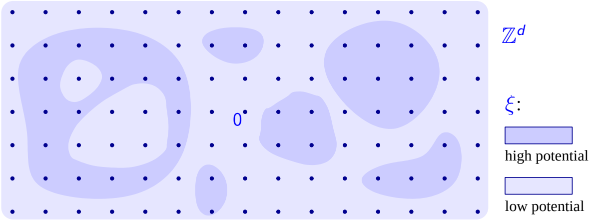

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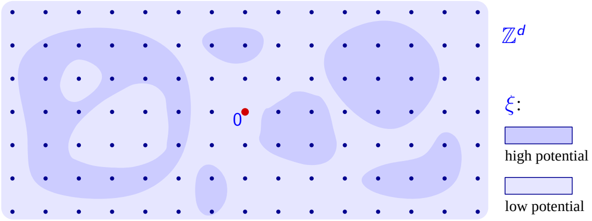

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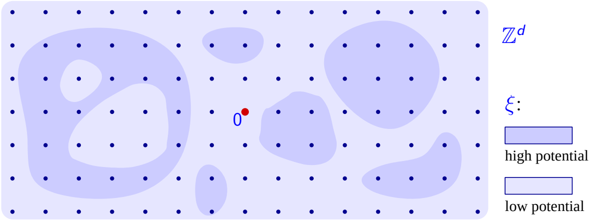

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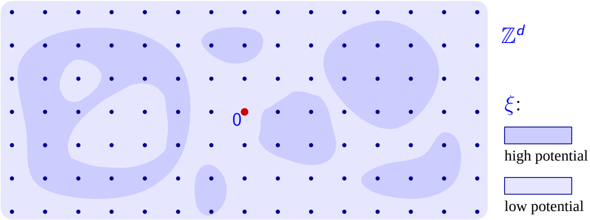

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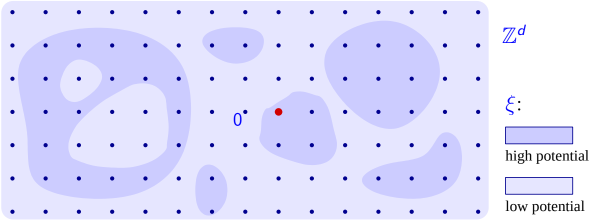

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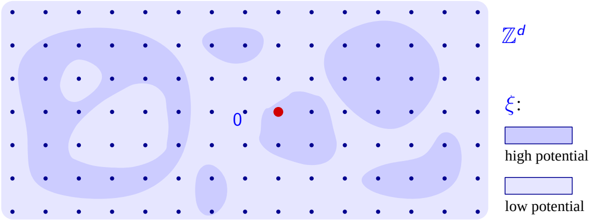

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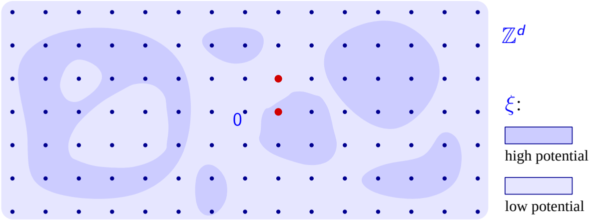

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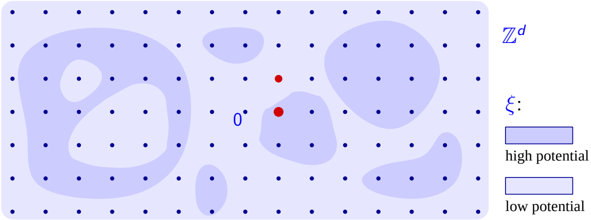

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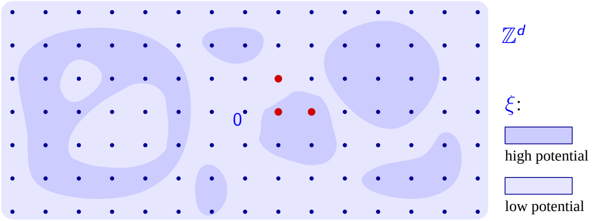

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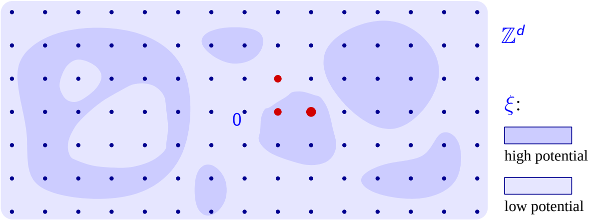

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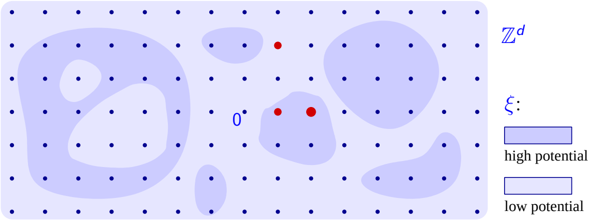

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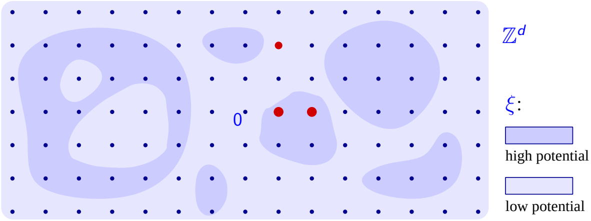

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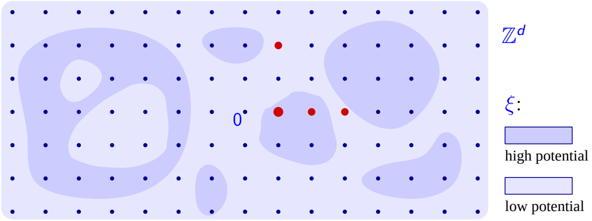

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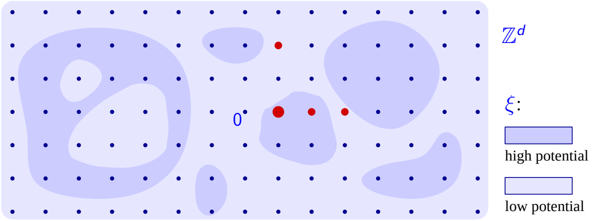

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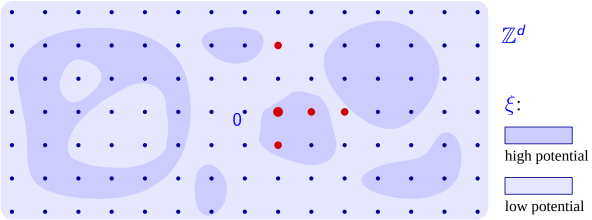

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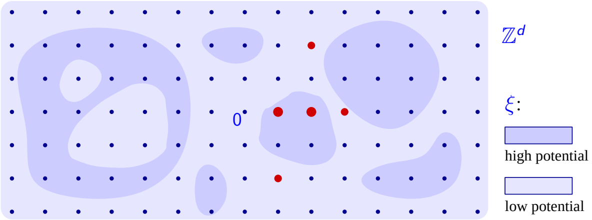

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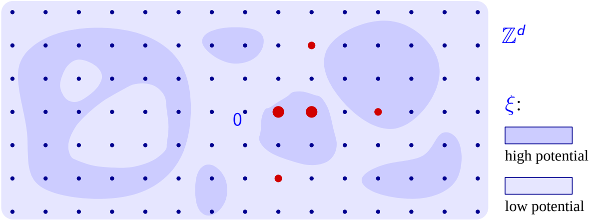

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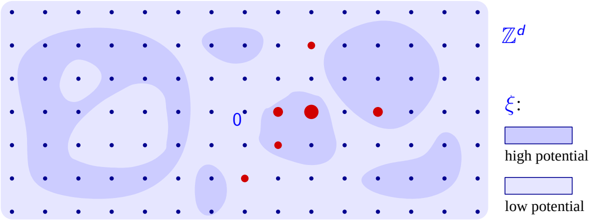

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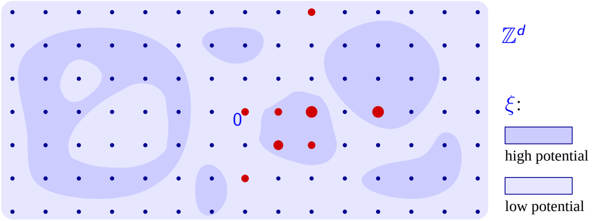

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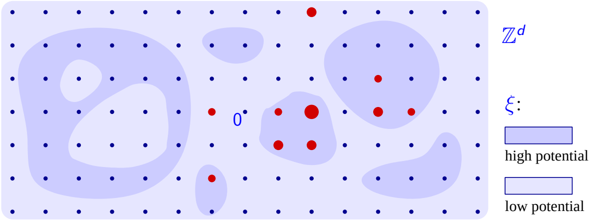

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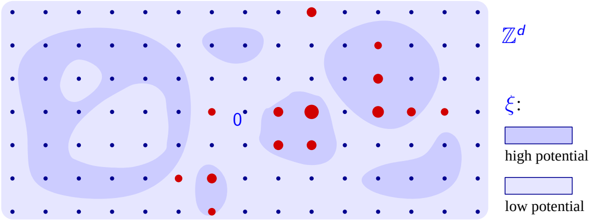

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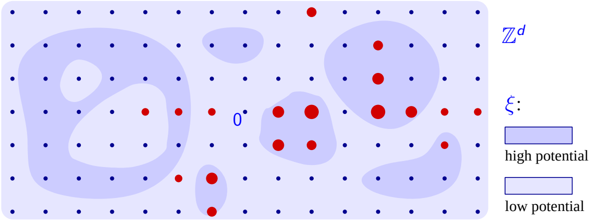

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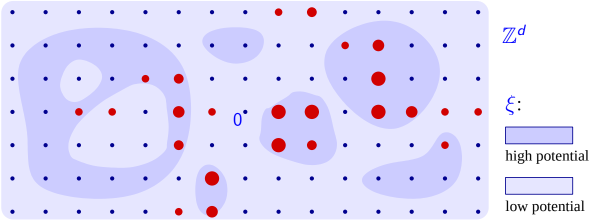

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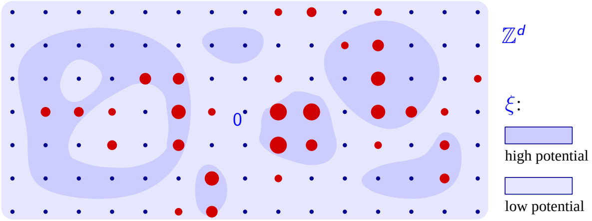

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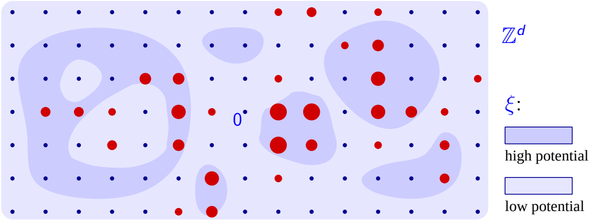


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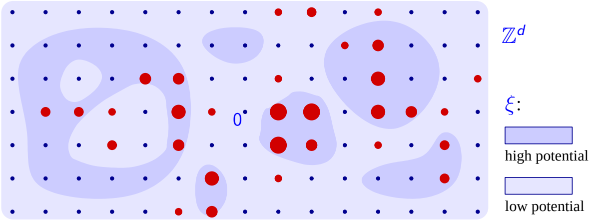


$N(t, z)$ is the number of particles at time $t$ at site $z$.

$u(t, z) = \mathbb{E}N(t, z)$ is the average number of particles at time $t$ at site $z$, still random.

# Two approaches to study $u(t, z)$

- Analytical:

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- Analytical: use Spectral Theory to analyse the parabolic Anderson equation

$$\frac{\partial u}{\partial t} = \Delta u + \xi u$$

- Probabilistic: use path analysis to analyse the Feynman–Kac Formula

$$u(t, z) = \mathbb{E}\Big\{ e^{\int_0^t \xi(X_s)\,\mathrm{d}s} \mathbf{1}_{\{X_t = z\}} \Big\},$$

where $(X_s)$ is a continuous-time random walk starting at zero.

# Heat equation

The propagation of temperature $u(t,x)$ at time $t$ at the point $x \in \mathbb{R}$ is described by

$$\frac{\partial u}{\partial t} = \Delta u.$$

If the initial temperature is $\delta_0$ then

$$u(t,x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}.$$

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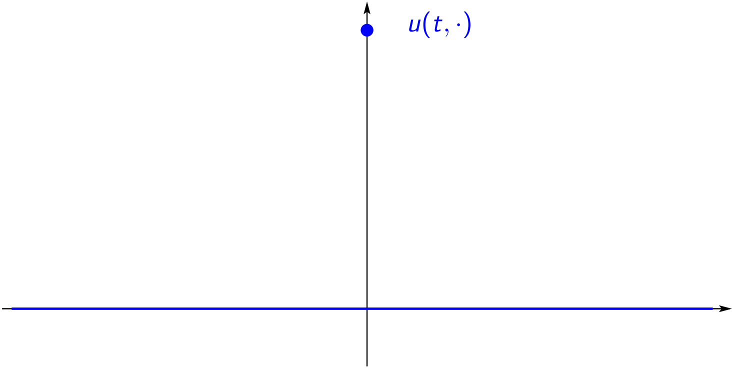

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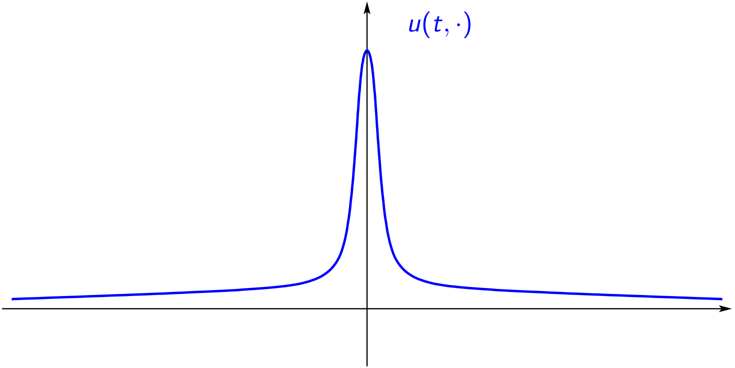

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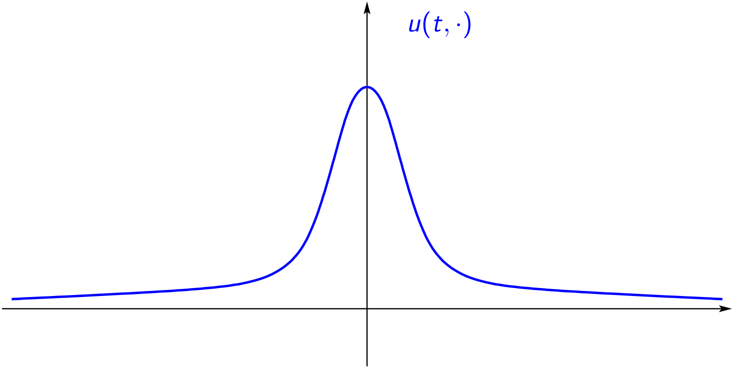

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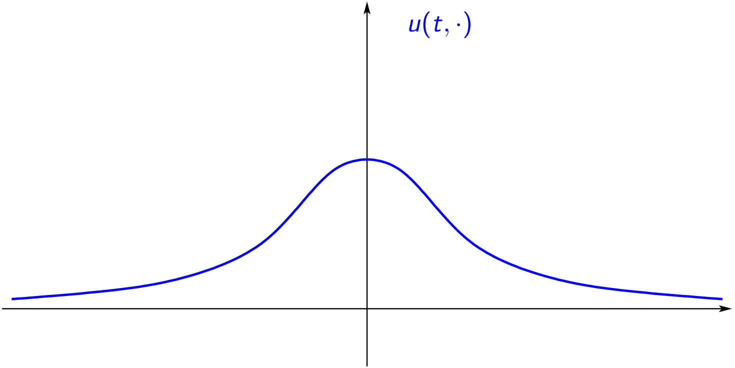

# Heat equation with a potential

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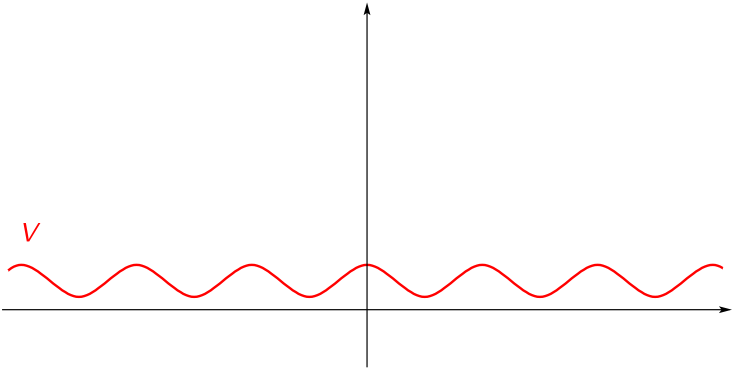
$$\frac{\partial u}{\partial t} = \Delta u + V u,$$

where $V : \mathbb{R} \to \mathbb{R}$ is a reasonably nice potential.



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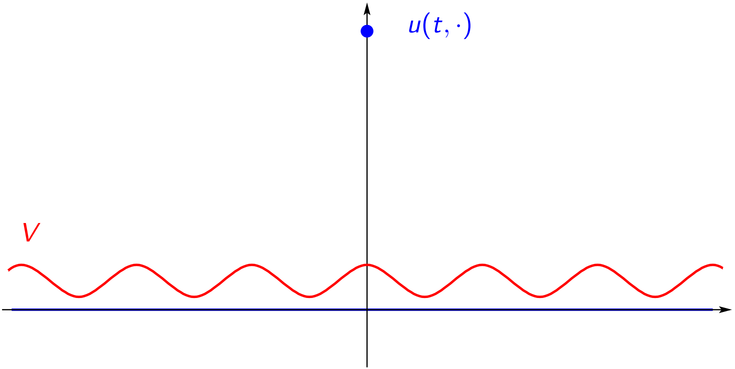

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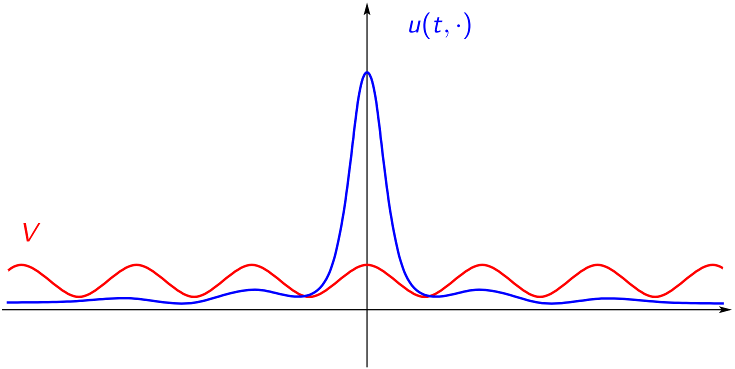

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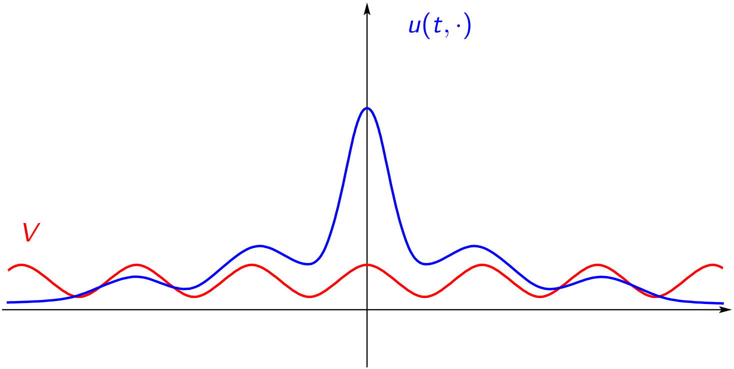

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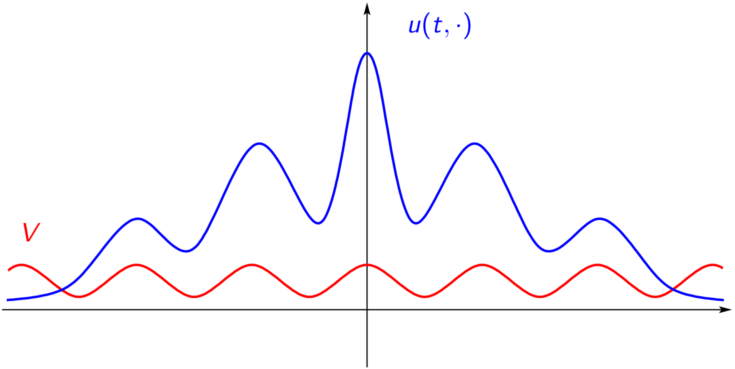

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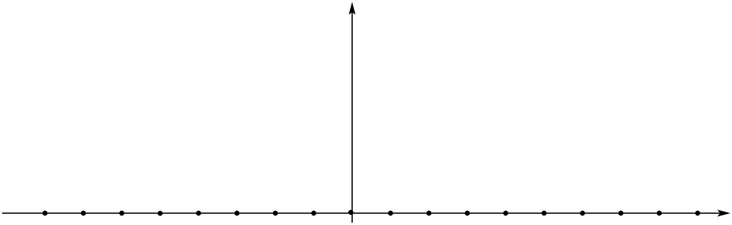

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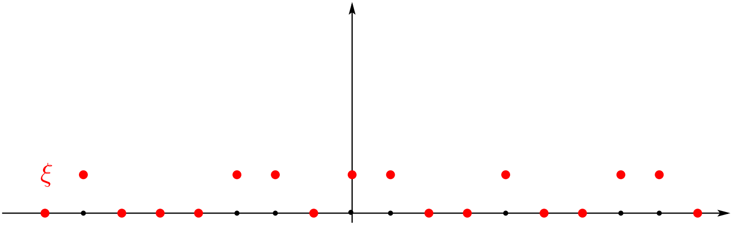
where $\xi : \mathbb{Z}^d \to \mathbb{R}$ is a random i.i.d. potential.



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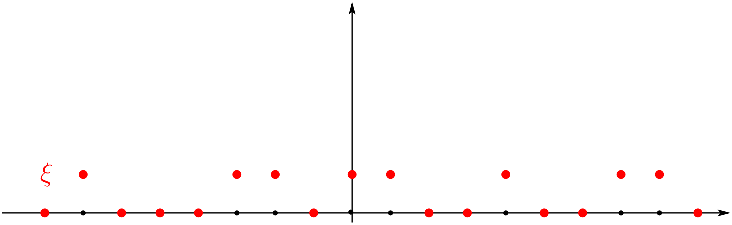


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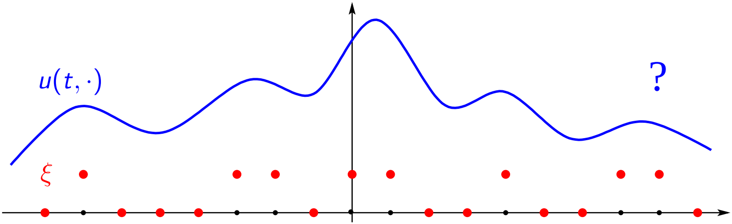


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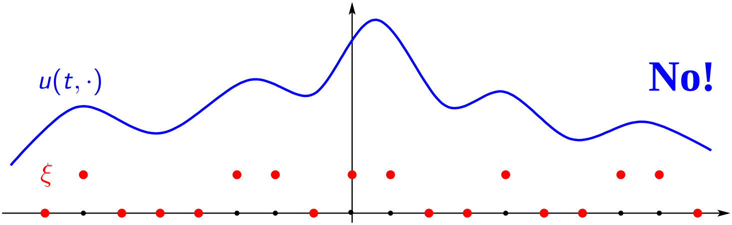


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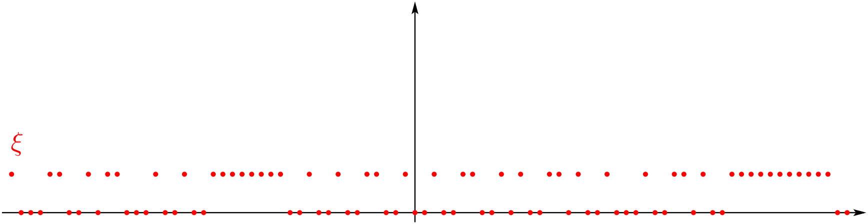

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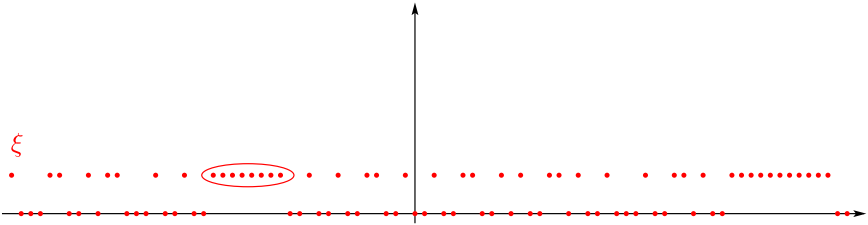

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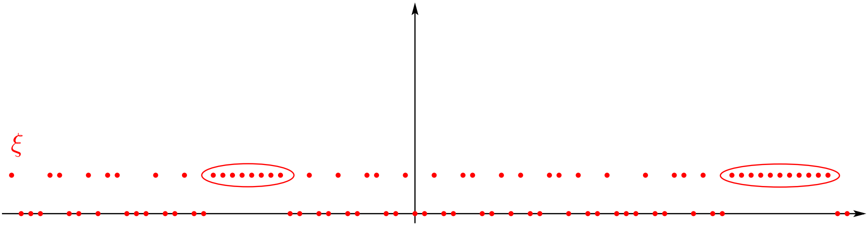

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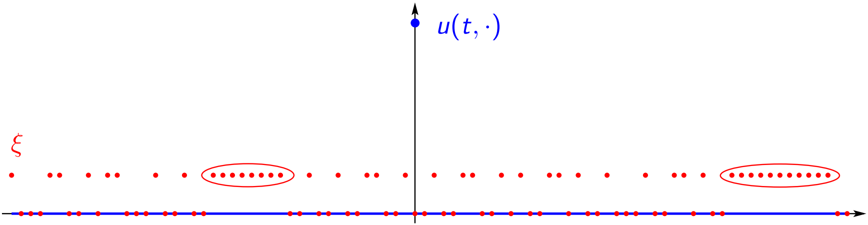

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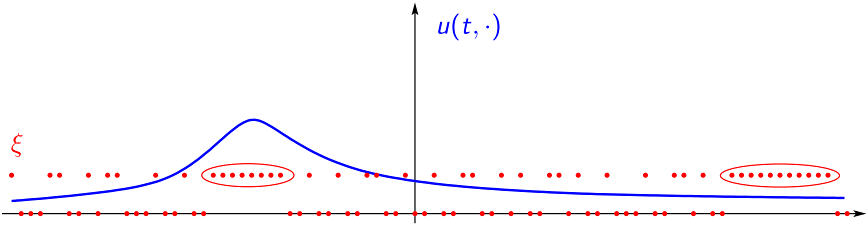

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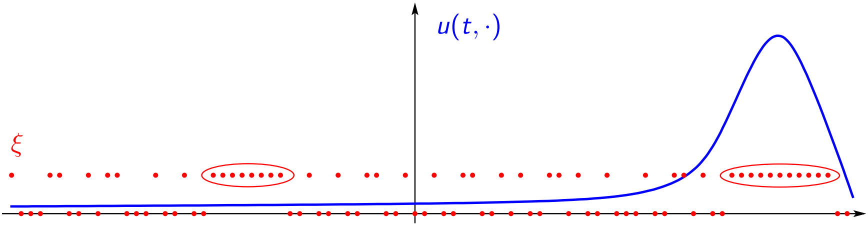

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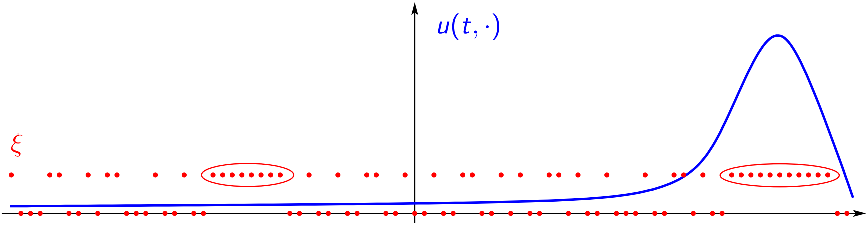


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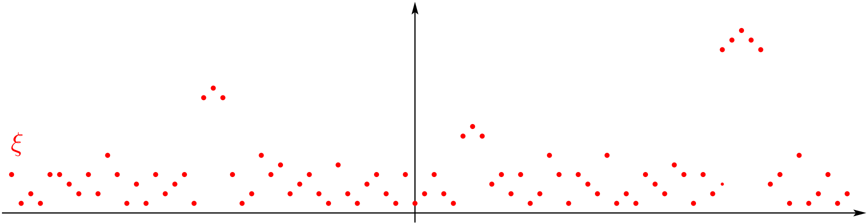

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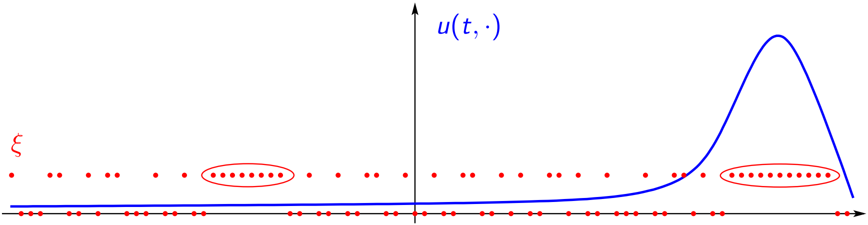


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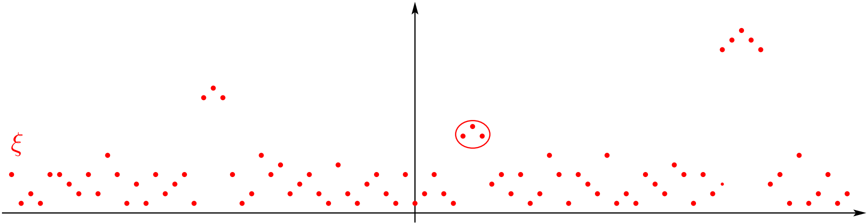

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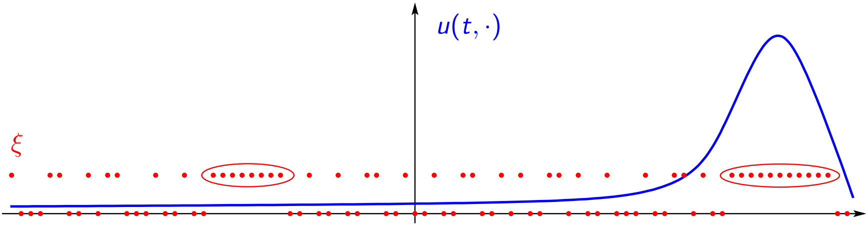


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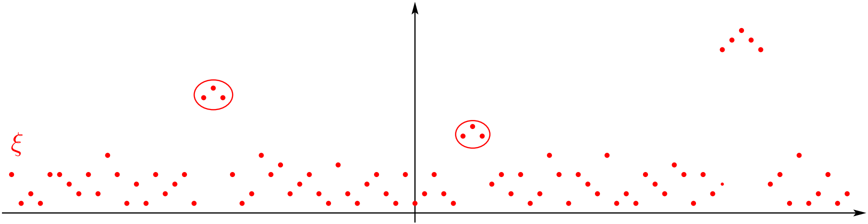

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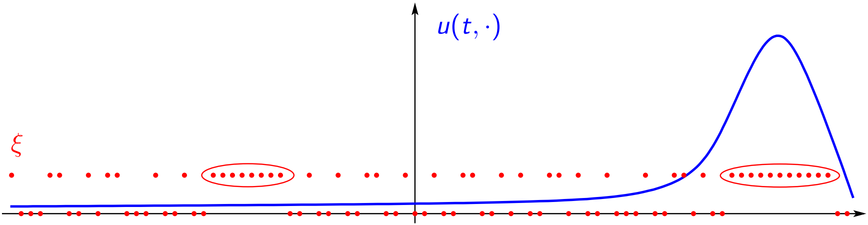


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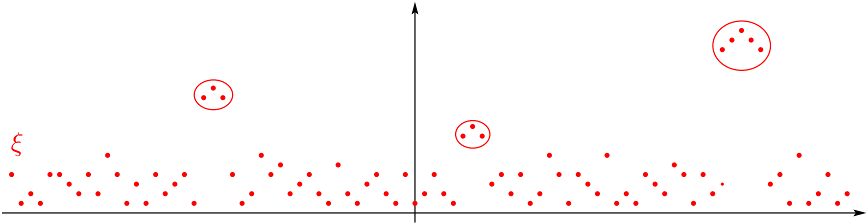

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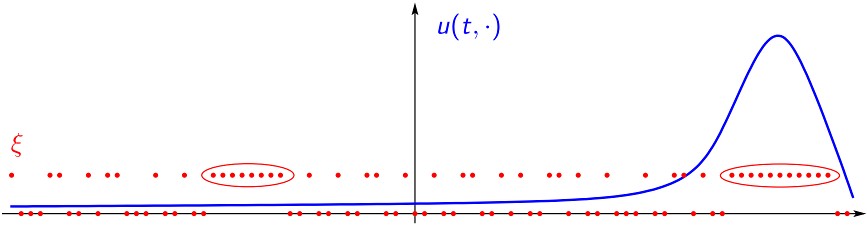


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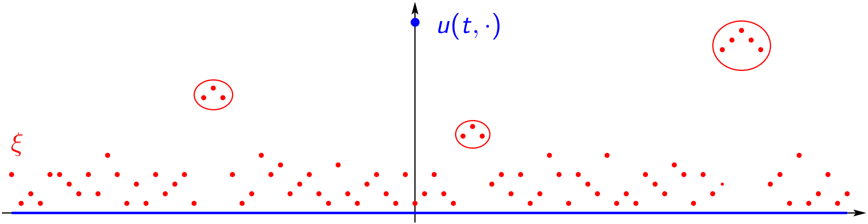

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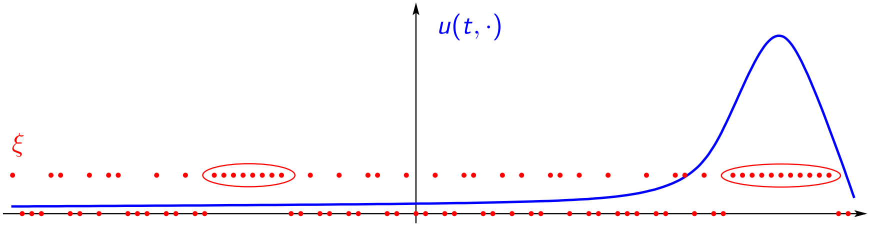


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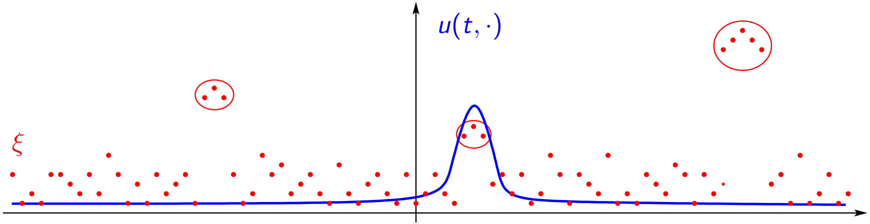

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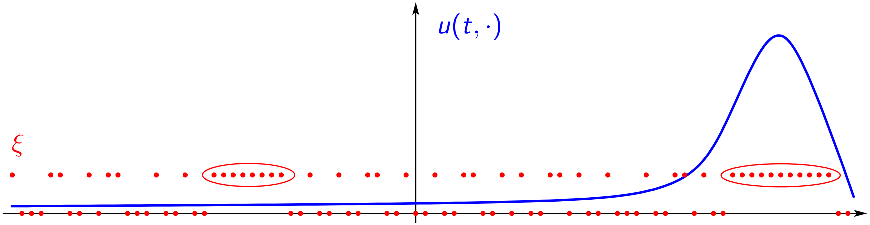


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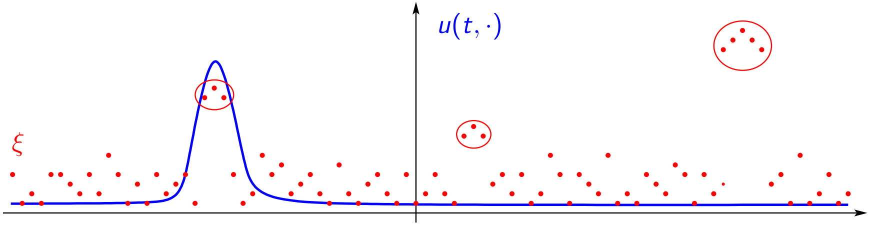

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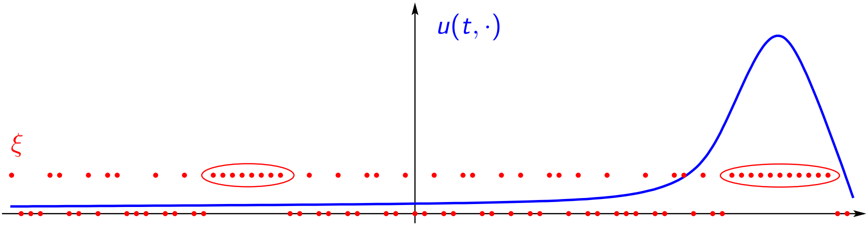


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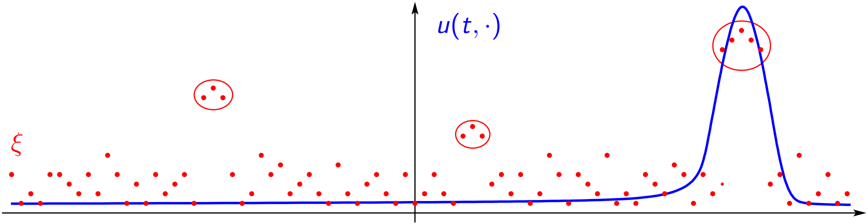

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What can we say about unbounded potentials?

- Pareto: $P(\xi(0) > x) = x^{-\alpha}$, $\alpha > d$
- Weibull: $P(\xi(0) > x) = \exp\{-x^{\gamma}\}$, $\gamma > 0$
- Double-exponential: $P(\xi(0) > x) = \exp\{-e^{x/\rho}\}$, $\rho > 0$
- 'Almost bounded' — quite different, not in this talk

# Localisation

**Theorem 1**

[König, Mörters, S. '06] Pareto
[S., Twarowski '12] Weibull with $\gamma < 2$
[Fiodorov, Muirhead '13] Weibull with any $\gamma$

There exists a process $Z_t$ with values in $\mathbb{Z}^d$ such that

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- For double-exponential potentials the solution $u(t, \cdot)$ is concentrated on one bounded ball. [Biskup, König, dos Santos, '16]

# Compliete localisation: localisation site

Theorem 2

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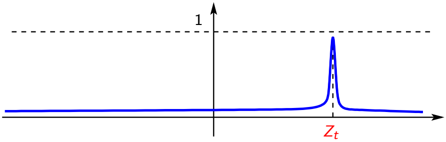
$$r_t = \begin{cases} \left(\frac{t}{\log t}\right)^{\frac{\alpha}{\alpha - d}} & \text{in the Pareto case,} \\ \frac{t(\log t)^{1/\gamma - 1}}{\log \log t} & \text{in the Weibull case.} \end{cases}$$

# Ageing



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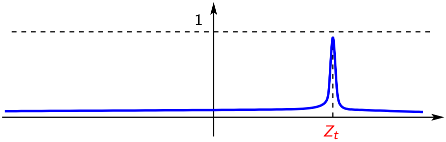
Waiting time until next change of state:

$$T_t = \inf\{s : Z_{t+s} \neq Z_t\}$$



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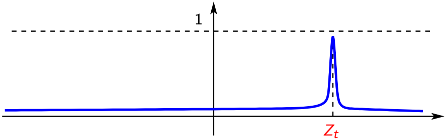


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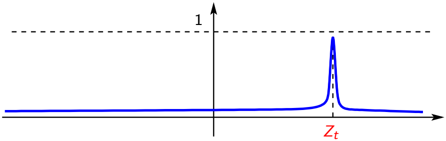


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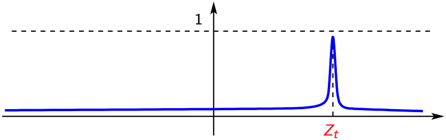


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where $\Theta$ is a nongegenerate a.s. positive random variable known explicitly.

The scaling limit of the whole process $(Z_t)$ can be described in terms of a Poisson point process [Mörters, Ortgiese, S.'11, Pareto].

# Almost sure localisation

Recall that (for Pareto and Weibull) we have

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**Theorem 4**

[König, Lacoin, Mörters, S.] Pareto:

There exist two processes $Z_t$ and $\hat{Z}_t$ with values in $\mathbb{Z}^d$ such that

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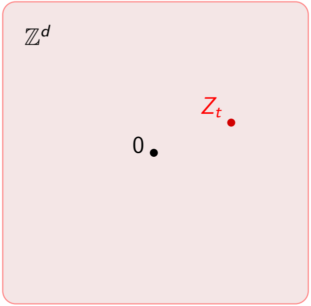
In the Weibull case this is an open question but is likely to be true.

# Idea of the proof of complete localisation

$$u(t,z) = \mathbb{E}\left\{ e^{\int_0^t \xi(X_s)\mathrm{d}s} \mathbf{1}_{\{\text{all paths from 0 to } z\}} \right\}$$



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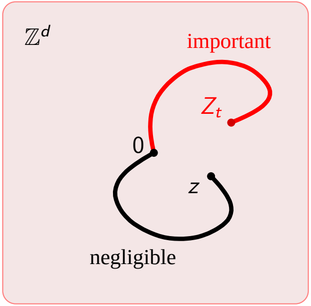

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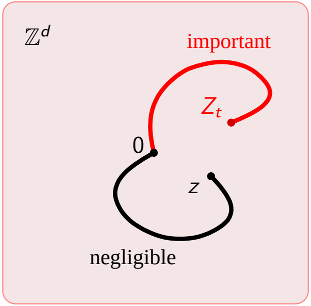

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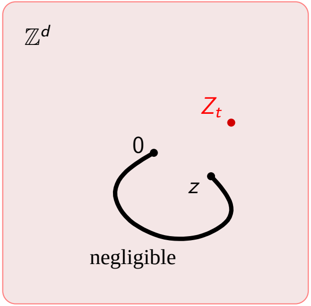

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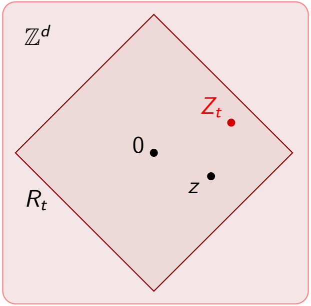

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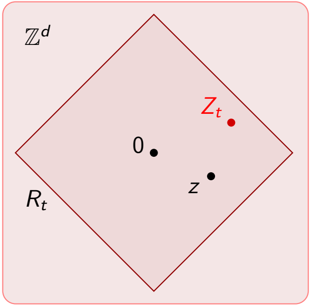

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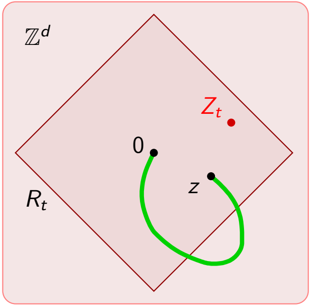
- $u_1(t,z) = \mathbb{E}\left\{ e^{\int_0^t \xi(X_s)\mathrm{d}s} \mathbf{1}_{\{\text{green paths}\}} \right\}$



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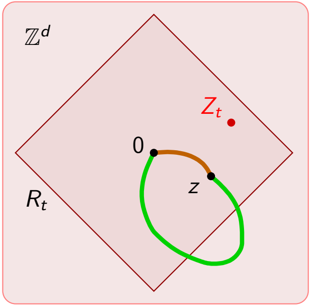

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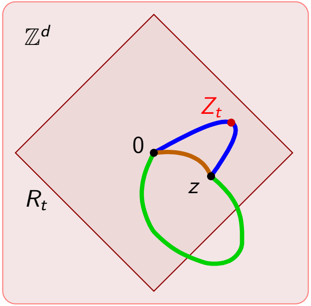

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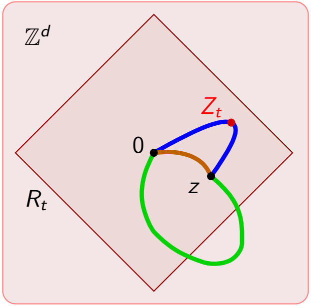


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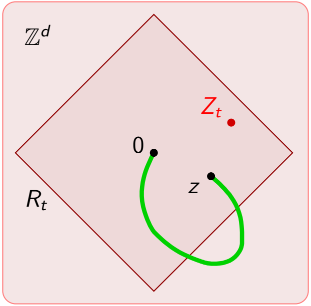


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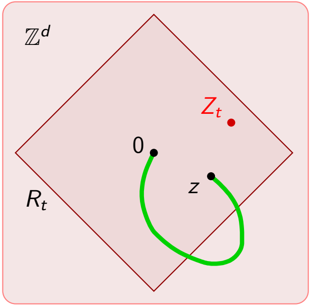


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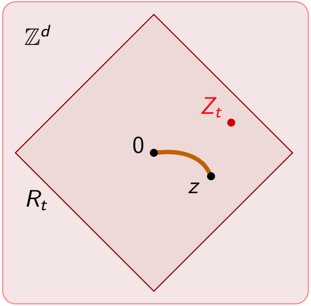


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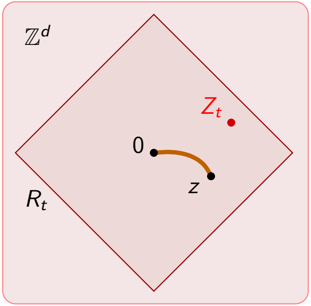


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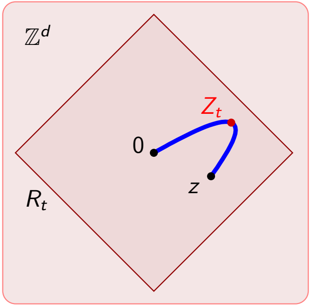


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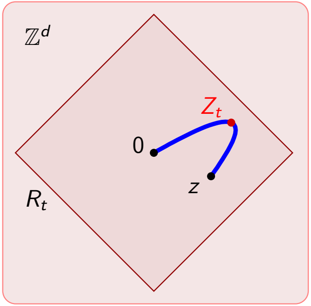


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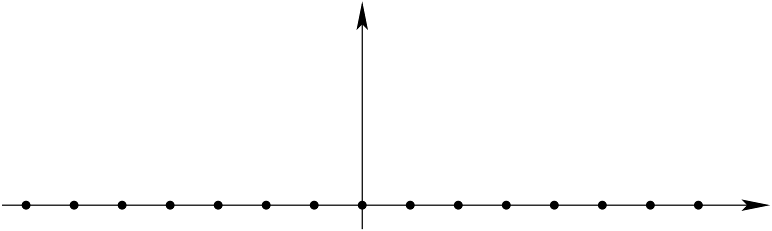

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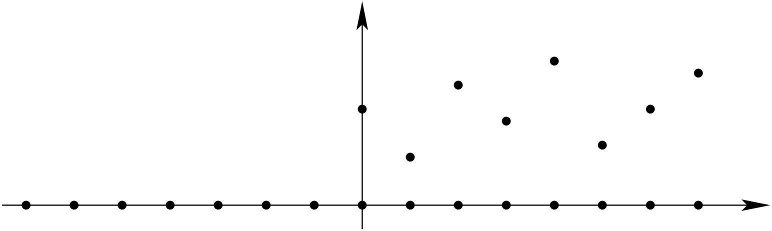

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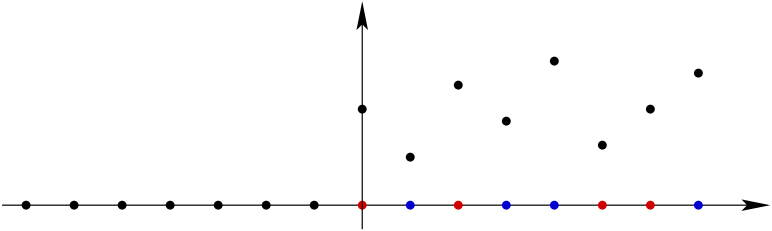

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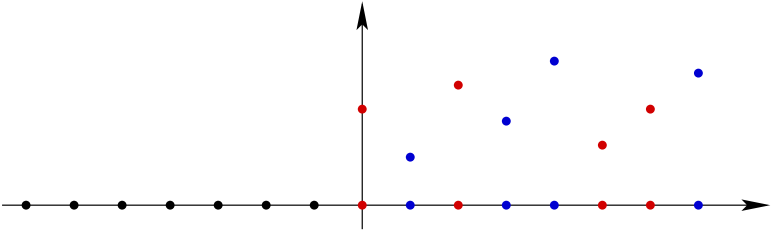

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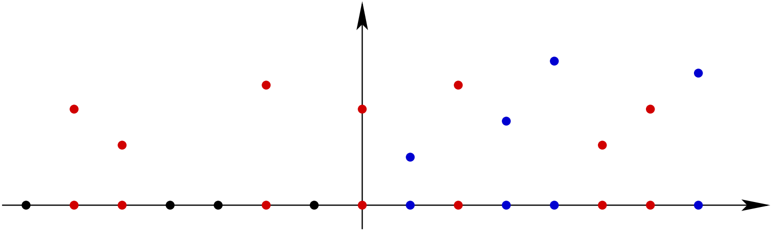

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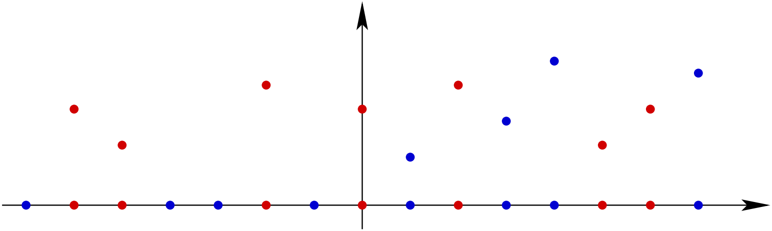

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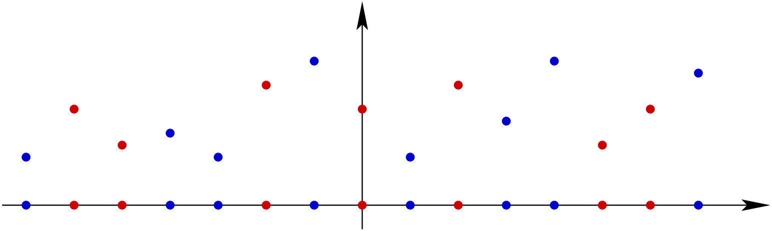

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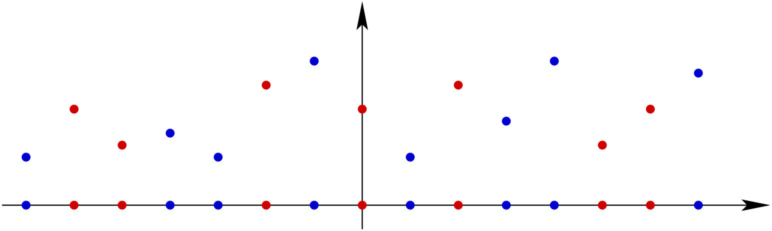

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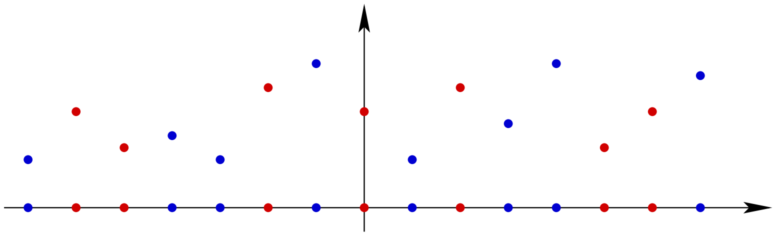

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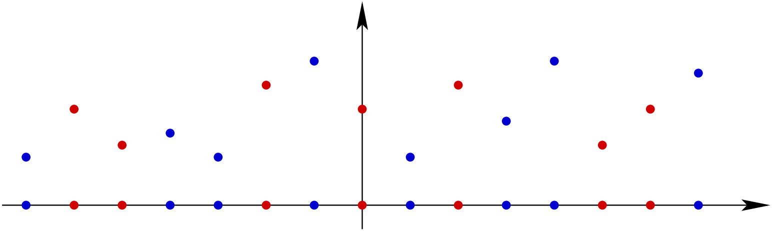

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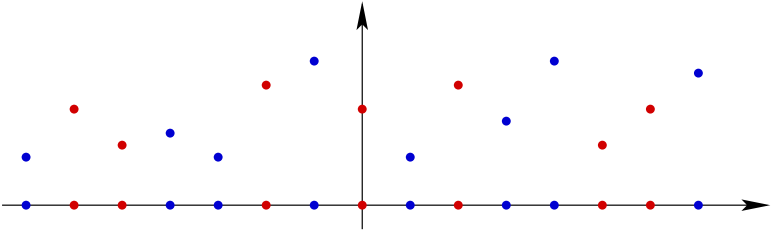

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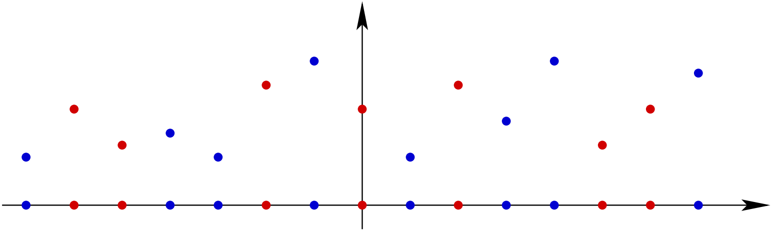

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# Answers

Theorem 1 (Muirhead, Pymar, S. '16)

Let $1 < \alpha < 2$.

Theorem 2 (Muirhead, Pymar, S. '16)

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Let $1 < \alpha < 2$.

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Let $\alpha \geq 2$. As $t \to \infty$, $$\frac{u(t, Z_t)}{U(t)} \to 1 \qquad \text{in probability.}$$ one point

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Let $\alpha \geq 2$, denote $q(n) = 1 - p(n)$, and introduce the critical scale

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where $\mathcal{P}_t^{\pm}$ are the sets of paths on $\mathbb{Z}$ starting at 0 and ending at $\pm Z_t$.

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In particular, for $1 < \alpha < 2$ we have

$$u(t, \pm Z_t) \sim e^{t\xi(Z_t) - 2t} \prod_{k=0}^{Z_t} \frac{1}{\xi(Z_t) - \xi(\pm k)}$$

# $1 < \alpha < 2$

$$\frac{u(t,Z_t)}{u(t,-Z_t)} \sim \prod_{k=1}^{Z_t} \frac{1}{\xi(Z_t)-\xi(k)} : \prod_{k=1}^{Z_t} \frac{1}{\xi(Z_t)-\xi(-k)}$$

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$$\sum_{i=1}^{n} X_i \approx n\mu + n^{1/2}\mathcal{N}$$

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$$\uparrow$$
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$$-\sum_{k:\text{non-dupl}} \log\Big(1-\frac{\xi(\pm k)}{\xi(Z_t)}\Big) \approx \frac{1}{\xi(Z_t)} \sum_{k:\text{non-dupl}} \xi(\pm k) \approx$$

# $1 < \alpha < 2$

$$\sum_{i=1}^{n} X_i \approx n\mu + n\mathcal{N}$$

$\uparrow$ normal — Pareto($\alpha$)

$$\frac{u(t,Z_t)}{u(t,-Z_t)} \sim \prod_{k=1}^{Z_t} \frac{1}{\xi(Z_t)-\xi(k)} : \prod_{k=1}^{Z_t} \frac{1}{\xi(Z_t)-\xi(-k)}$$

$$= \exp\Big\{ - \sum_{k:\text{non-dupl}} \log\Big(1-\frac{\xi(k)}{\xi(Z_t)}\Big) + \sum_{k:\text{non-dupl}} \log\Big(1-\frac{\xi(-k)}{\xi(Z_t)}\Big)\Big\}$$

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# $1 < \alpha < 2$

$$\sum_{i=1}^{n} X_i \approx n\mu + n^{1/\alpha}\mathcal{N}$$

$\uparrow$ stable Pareto($\alpha$)

$$\frac{u(t,Z_t)}{u(t,-Z_t)} \sim \prod_{k=1}^{Z_t} \frac{1}{\xi(Z_t)-\xi(k)} : \prod_{k=1}^{Z_t} \frac{1}{\xi(Z_t)-\xi(-k)}$$
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same for $\pm Z_t$

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Hence everything is determined by

$$\frac{|Z_t|^{1/\alpha}}{\xi(Z_t)} \asymp$$

# $1 < \alpha < 2$

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Hence everything is determined by

$$\frac{|Z_t|^{1/\alpha}}{\xi(Z_t)} \asymp \begin{cases} \infty & \Rightarrow \text{one point dominates} \\ 1 & \Rightarrow \text{random proportion of mass at each point} \\ 0 & \Rightarrow 1/2 \text{ of the mass at each point} \end{cases}$$

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$$\frac{|Z_t|^{1/\alpha}}{\xi(Z_t)} \asymp \frac{r_t^{1/\alpha}}{a_t} \asymp \begin{cases} \infty & \Rightarrow \text{one point dominates} \\ 1 & \Rightarrow \text{random proportion of mass at each point} \\ 0 & \Rightarrow 1/2 \text{ of the mass at each point} \end{cases}$$

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Miracle! $a_t = r_t^{1/\alpha}$

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The scale of fluctuations remains finite for all values of $1 < \alpha < 2$.

# $1 < \alpha < 2$

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stable ↑ Pareto($\alpha$)

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The scale of fluctuations remains finite for all values of $1 < \alpha < 2$.