Hybrid discontinuous Galerkin discretisations and domain decomposition preconditioners for the Stokes problem

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Outline

Motivation

Hybrid discontinuous Galerkin (HDG) methods for Stokes

Domain decomposition methods

Two-level methods

Conclusion
Outline

Motivation

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Conclusion
Symbolic approaches for systems of PDEs

Motivation:

- extend well-known DD preconditioners for scalar PDEs to discretised systems of PDEs.
- examples: Stokes/Oseen equations (as the first step before dealing with the difficulties of the numerical simulation of the Navier-Stokes equations) and linear elasticity.

Idea:

- DD preconditioners involve the solution of local (smaller) boundary value problems (BVP) for which the boundary conditions can influence the convergence.
- use symbolic algebraic tools to derive (formally) the best boundary conditions for the local BVP leading to an optimal convergence for a given system.
Stokes with non standard boundary conditions

Algebraic tools lead to non-standard BVP to be solved

TVNF (tangential velocity-normal flux)

\[
\begin{align*}
-\nu \Delta u + \nabla p &= f \text{ in } \Omega \\
\nabla \cdot u &= 0 \text{ in } \Omega \\
\sigma_{nn} &= g \text{ on } \Gamma \\
u_t &= 0 \text{ on } \Gamma \\
\end{align*}
\]

NVTF (normal velocity-tangential flux)

\[
\begin{align*}
-\nu \Delta u + \nabla p &= f \text{ in } \Omega \\
\nabla \cdot u &= 0 \text{ in } \Omega \\
\sigma_{nt} &= g \text{ on } \Gamma \\
u_n &= 0 \text{ on } \Gamma \\
\end{align*}
\]


Notations and questions

- $\Omega \subset \mathbb{R}^2$ is open polygonal domain with Lipschitz boundary $\Gamma := \partial \Omega$ and $n$ the outward normal to $\Gamma$.
- $u \in [H^1(\Omega)]^2$ is the velocity vector field, $p \in L^2(\Omega)$ is the pressure.
- $f \in [L^2(\Omega)]^2$, $g \in L^2(\Omega)$ are given functions.
- $\nu = \text{const}$ is the viscosity.
- $\sigma := \nu \nabla u - pl$ is the stress tensor, $\sigma_n := \sigma \cdot n$ is the flux.
- $u_n := u \cdot n$, $\sigma_{nn} := \sigma_n \cdot n$ are the normal components.
- $u_t := u \cdot t$, $\sigma_{nt} := \sigma_n \cdot t$ are the tangential components.

Question: how to discretise the non-standard BVP such that the boundary conditions should be naturally taken into account?
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Finite elements methods for Stokes

Continuous

Discontinuous

$H(\text{div})$-conforming

Hybrid discontinuous

$H(\text{div})$-conforming
Discrete spaces

- $\mathcal{T}_h$ is the collection of disjoint triangles $K$ that are partition of $\Omega$
- $\mathcal{E}_h$ is the collection of all the edges
- $h_K$ is the diameter of the triangle $K$ and $h := \max_{K \in \mathcal{T}_h} h_K$

$$BDM_{h,0}^k := \left\{ \mathbf{v}_h \in H(\text{div}, \Omega) : \mathbf{v}_h|_K \in [\mathbb{P}_k (K)]^2 \ \forall \ K \in \mathcal{T}_h \land (\mathbf{v}_h)_n = 0 \text{ on } \Gamma \right\}$$

$$Q_{h,0}^{k-1} := \left\{ q_h \in L^2(\Omega) : q_h|_K \in \mathbb{P}_{k-1} (K) \ \forall \ K \in \mathcal{T}_h \land \int_{\Omega} q_h \, dx = 0 \right\}$$

$$M_{h,0}^{k-1} := \left\{ \tilde{v}_h \in L^2(\mathcal{E}_h) : \tilde{v}_h|_E \in \mathbb{P}_{k-1} (E) \ \forall \ E \in \mathcal{E}_h \land \tilde{v}_h = 0 \text{ on } \Gamma \right\}$$

$L^2$-projection

$$\Phi_{E}^{k-1} : L^2 (E) \rightarrow \mathbb{P}_{k-1} (E) \ \text{s. t.} \ \forall \ \tilde{v}_h \in \mathbb{P}_{k-1} (E)$$

$$\int_{E} \Phi_{E}^{k-1} (\tilde{w}_h) \tilde{v}_h \, ds = \int_{E} \tilde{w}_h \tilde{v}_h \, ds \quad \Phi_{E}^{k-1}|_E := \Phi_{E}^{k-1} \forall E \in \mathcal{E}_h$$
Discretisation

\[-\nu \Delta u + \nabla p = f \quad \text{in } \Omega\]

\[\Downarrow\]

\[\sum_{K \in \mathcal{T}_h} \left( \int_K \nu \nabla u : \nabla v_h \, dx - \int_K p \nabla \cdot v_h \, dx \right) - \int_{\partial K} \nu \partial_n u \, v_h \, ds + \int_{\partial K} p(v_h)_n \, ds = \int_{\Omega} f v_h \, dx\]
Discretisation

\[-\nu \Delta u + \nabla p = f \quad \text{in } \Omega\]

\[
\sum_{K \in T_h} \left( \int_K \nu \nabla u : \nabla v_h \, dx - \int_K p \nabla \cdot v_h \, dx \right.
\]

\[
- \int_{\partial K} \nu (\partial_n u)_t (v_h)_t \, ds \left( 
- \int_{\Gamma} \sigma_{nn} (v_h)_n \, ds = \int_{\Omega} f v_h \, dx \right)
\]
Discretisation

\[-\nu \Delta u + \nabla p = f \quad \text{in } \Omega\]

\[\downarrow\]

\[
\sum_{K \in \mathcal{T}_h} \left( \int_K \nu \nabla u : \nabla \mathbf{v}_h \, d\mathbf{x} - \int_K p \nabla \cdot \mathbf{v}_h \, d\mathbf{x} \\
- \int_{\partial K} \nu (\partial_n u)_t ((\mathbf{v}_h)_t - \tilde{\mathbf{v}}_h) \, ds\right)
\]

\[-\int_{\Gamma} \sigma_{nn}(\mathbf{v}_h)_n \, ds - \int_{\Gamma} \sigma_{nt} \tilde{\mathbf{v}}_h \, ds = \int_{\Omega} f \mathbf{v}_h \, d\mathbf{x}\]
Discretisation

\[-\nu \Delta u + \nabla p = f \quad \text{in } \Omega\]

\[\downarrow\]

\[
\sum_{K \in \mathcal{T}_h} \left( \int_K \nu \nabla u : \nabla v_h \, dx - \int_K p \nabla \cdot v_h \, dx \right. \\
\left. - \int_{\partial K} \nu \left( \partial_n u \right)_t \left( (v_h)_t - \tilde{v}_h \right) \, ds \right.
\]

\[+ \varepsilon \int_{\partial K} \nu (u_t - \tilde{u}) \left( \partial_n v \right)_t \, ds \quad \tilde{u} := u_t \text{ on } \mathcal{E}_h
\]

\[+ \frac{\tau}{h_K} \int_{\partial K} \nu (u_t - \tilde{u}) (v_t - \tilde{v}) \, ds \right)
\]

\[- \int_{\Gamma} \sigma_{nn} (v_h)_n \, ds - \int_{\Gamma} \sigma_{nt} \tilde{v}_h \, ds = \int_{\Omega} f v_h \, dx\]
Bilinear forms

Velocity bilinear form

\[
a ( ( w_h, \tilde{w}_h ), ( v_h, \tilde{v}_h ) ) := \sum_{K \in T_h} \left( \int_K \nu \nabla w_h : \nabla v_h \ dx \right.
- \int_{\partial K} \nu ( \partial_n w_h )_t \left( ( v_h )_t - \tilde{v}_h \right) \ ds \\
+ \varepsilon \int_{\partial K} \nu ( ( w_h )_t - \tilde{w}_h ) ( \partial_n v_h )_t \ ds \\
+ \nu \frac{\tau}{h_K} \int_{\partial K} \Phi^{k-1} ( ( w_h )_t - \tilde{w}_h ) \Phi^{k-1} ( ( v_h )_t - \tilde{v}_h ) \ ds \right)
\]

Pressure bilinear form

\[
b ( ( v_h, \tilde{v}_h ), q_h ) := - \sum_{K \in T_h} \int_K q_h \nabla \cdot v_h \ dx
\]
Discrete problems

TVNF boundary value problem

Find \((u_h, \tilde{u}_h, p_h) \in BDM_h \times M_{h,0}^{k-1} \times Q_h^{k-1} =: \mathcal{V} \) s. t. \(\forall (v_h, \tilde{v}_h, q_h) \in \mathcal{V}\)

\[
\begin{align*}
 a ((u_h, \tilde{u}_h), (v_h, \tilde{v}_h)) & + b ((v_h, \tilde{v}_h), p_h) \quad = \quad \int_{\Omega} f v_h \, dx + \int_{\Gamma} g(v_h)_n \, ds \\
 b ((u_h, \tilde{u}_h), q_h) & \quad = \quad 0.
\end{align*}
\]

NVTF boundary value problem

Find \((u_h, \tilde{u}_h, p_h) \in BDM_{h,0} \times M_h^{k-1} \times Q_{h,0}^{k-1} =: \mathcal{W} \) s. t. \(\forall (v_h, \tilde{v}_h, q_h) \in \mathcal{W}\)

\[
\begin{align*}
 a ((u_h, \tilde{u}_h), (v_h, \tilde{v}_h)) & + b ((v_h, \tilde{v}_h), p_h) \quad = \quad \int_{\Omega} f v_h \, dx + \int_{\Gamma} g\tilde{v}_h \, ds \\
 b ((u_h, \tilde{u}_h), q_h) & \quad = \quad 0.
\end{align*}
\]

Similar approach but with discontinuous spaces (instead of \(H(div)\)) \(\rightarrow\) more degrees of freedom.


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Main results

We proved well-posedness with respect to the discrete norm

\[ \| (\mathbf{w}_h, \tilde{\mathbf{w}}_h) \|^2 := \nu \sum_{K \in T_h} \left( |\mathbf{w}_h|_{H^1(K)}^2 + h_K \| \partial_n \mathbf{w}_h \|_{\partial K}^2 \right) + \frac{\tau}{h_K} \| \Phi^{k-1} ( (\mathbf{w}_h)_t - \tilde{\mathbf{w}}_h ) \|_{\partial K}^2. \]

The \( h^2 \) optimal convergence of the velocity with respect to the \( L^2 \) norm \( \| \cdot \|_{\Omega} \) is confirmed numerically.

TVNF boundary value problem

NVTF boundary value problem

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Stabilised hybrid discontinuous Galerkin

\[ BDM_{h,0}^k := \left\{ v_h \in H(\text{div}, \Omega) : v_h|_K \in [\mathbb{P}_k(K)]^2 \ \forall \ K \in \mathcal{T}_h \land (v_h)_n = 0 \text{ on } \Gamma \right\} \]

\[ Q_{h,0}^k := \left\{ q_h \in L^2(\Omega) : q_h|_K \in \mathbb{P}_k(K) \ \forall \ K \in \mathcal{T}_h \land \int_{\Omega} q_h \, dx = 0 \right\} \]

\[ M_{h,0}^{k-1} := \left\{ \tilde{v}_h \in L^2(\mathcal{E}_h) : \tilde{v}_h|_E \in \mathbb{P}_{k-1}(E) \ \forall \ E \in \mathcal{E}_h \land \tilde{v}_h = 0 \text{ on } \Gamma \right\} \]

**$L^2$-projections**

\[ \Psi_{k}^{k-1} : L^2(K) \to \mathbb{P}_{k-1}(K) \text{ s. t. } \forall \ v_h \in \mathbb{P}_{k-1}(K) \]

\[ \int_K \Psi_{K}^{k-1}(w)v_h \, dx = \int_K wv_h \, dx \quad \Psi^{k-1}|_K := \Psi_{K}^{k-1} \ \forall \ K \in \mathcal{T}_h \]

\[ \Phi_{E}^{k-1} : L^2(E) \to \mathbb{P}_{k-1}(E) \text{ s. t. } \forall \ \tilde{v}_h \in \mathbb{P}_{k-1}(E) \]

\[ \int_E \Phi_{E}^{k-1}(\tilde{w}_h)\tilde{v}_h \, ds = \int_E \tilde{w}_h\tilde{v}_h \, ds \quad \Phi^{k-1}|_E := \Phi_{E}^{k-1} \ \forall \ E \in \mathcal{E}_h \]
Stabilised hybrid discontinuous Galerkin

TVNF boundary value problem
Find \((u_h, \tilde{u}_h, p_h) \in BDM_h \times M_{h,0}^{k-1} \times Q_h^k =: \mathcal{V}\) s. t. \(\forall (v_h, \tilde{v}_h, q_h) \in \mathcal{V}\)

\[
\begin{cases}
a ((u_h, \tilde{u}_h), (v_h, \tilde{v}_h)) + b ((v_h, \tilde{v}_h), p_h) &= \int_{\Omega} f v_h \, dx + \int_{\Gamma} g(v_h)_n \, ds \\
b ((u_h, \tilde{u}_h), q_h) - s (p_h, q_h) &= 0,
\end{cases}
\]

NVTF boundary value problem
Find \((u_h, \tilde{u}_h, p_h) \in BDM_{h,0} \times M_{h}^{k-1} \times Q_{h,0}^k =: \mathcal{W}\) s. t. \(\forall (v_h, \tilde{v}_h, q_h) \in \mathcal{W}\)

\[
\begin{cases}
a ((u_h, \tilde{u}_h), (v_h, \tilde{v}_h)) + b ((v_h, \tilde{v}_h), p_h) &= \int_{\Omega} f v_h \, dx + \int_{\Gamma} g\tilde{v}_h \, ds \\
b ((u_h, \tilde{u}_h), q_h) - s (p_h, q_h) &= 0,
\end{cases}
\]

\[s (p_h, q_h) := \frac{1}{\nu} \int_{\Omega} (p_h - \psi^{k-1} p_h) (q_h - \psi^{k-1} q_h) \, dx.\]
Main results

We proved the well-posedness and convergence with respect to the norm

$$\|(w_h, \tilde{w}_h, q_h)\|_h := \|(w_h, \tilde{w}_h)\| + \frac{1}{\nu} \|q_h\|_\Omega.$$  

We observed the $h^2$ optimal convergence of the velocity with respect to the $L^2$ norm $\|\cdot\|_\Omega$. 

TVNF boundary value problem  

NVTF boundary value problem
Outline

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Domain decomposition methods

Two-level methods

Conclusion
Domain decomposition preconditioning

Linear systems solvers:

- **direct solvers** (e.g. LU): robust, but high memory cost, difficult to parallelize,
- **iterative solvers** (e.g. CG, GMRES): low memory cost, easy to parallelize, but not robust.

⇒ **Domain Decomposition (DD) preconditioner** for the iterative solver (GMRES)

DD methods: suited to parallel computing by construction and subproblems of a smaller dimension.

Domain $\Omega$ decomposed into $N_{\text{sub}}$ overlapping **subdomains** $\Omega_i$

(mesh partitioner SCOTCH and add layers of overlap)
Overlapping domain decomposition methods

Consider the linear system: $Au = f \in \mathbb{C}^n$.

Given a decomposition of $[1; n]$, $(\mathcal{N}_1, \mathcal{N}_2)$, define:

- the restriction operator $R_i$ from $[1; n]$ into $\mathcal{N}_i$,
- $R_i^T$ as the extension by 0 from $\mathcal{N}_i$ into $[1; n]$.

Duplicated unknowns coupled via a partition of unity:

$$I = \sum_{i=1}^{N} R_i^T D_i R_i.$$

- $M^{-1}_{RAS} := \sum_{i=1}^{N} R_i^T D_i A_i^{-1} R_i$ with $A_i = R_i A R_i^T$
- $M^{-1}_{MRAS} := \sum_{i=1}^{N} R_i^T D_i B_i^{-1} R_i$ Modified Restricted Additive Schwarz
Numerical experiments for two dimensional problems

We compared the newly introduced preconditioners to the more standard restricted additive Schwarz preconditioner. Numerical experiments clearly show their superiority for different test cases in two space dimensions.

![Taylor-Hood discretisation](chart1.png)

![hdG discretisation](chart2.png)


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Two level domain decomposition preconditioners

One level methods are not scalable with respect to the number of subdomains.

Taylor-Hood discretisation

hdG discretisation
Two-level methods

Versions of the one-level preconditioner

- The MRAS preconditioner

\[
M_{\text{MRAS}}^{-1} = \sum_{i=1}^{N} R_i^T D_i B_i^{-1} R_i
\]

- Symmetrised version of the MRAS preconditioner

\[
M_{\text{SMRAS}}^{-1} = \sum_{i=1}^{N} R_i^T D_i B_i^{-1} D_i R_i
\]

\(B_i\) are local matrices associated to a discretisation of local TVNF or NVTF boundary value problem.
Two-level methods

▶ Versions of the two-level preconditioner

\[
M_{SMRAS,2}^{-1} = P_0 A^{-1} + (\text{Id} - P_0) \left( \sum_{i=1}^{N} R_i^T D_i B_i^{-1} D_i R_i \right) (\text{Id} - P_0^T)
\]

\[
M_{MRAS,2}^{-1} = P_0 A^{-1} + (\text{Id} - P_0) \left( \sum_{i=1}^{N} R_i^T D_i B_i^{-1} R_i \right) (\text{Id} - P_0^T)
\]

where \( P_0 \) is \( A \)-orthogonal projection onto the coarse space associated with the following generalised eigenvalue problem

▶ Generalised eigenvalue problem: Find \((V_{jk}, \lambda_{jk}) \in \mathbb{R}^{\vert N_j \vert} \setminus \{0\} \times \mathbb{R}\) s. t.

\[
\tilde{A}_j V_{jk} = \lambda_{jk} B_j V_{jk},
\]

where \( \tilde{A}_j \) are local matrices associated to a discretisation of local Neumann boundary value problem in \( \Omega_i \).


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Model boundary value problems

- Stokes equations

\[
\begin{align*}
-\nu \Delta u + \nabla p &= f \quad \text{in } \Omega \\
- \nabla \cdot u &= 0 \quad \text{in } \Omega
\end{align*}
\]

- Nearly incompressible elasticity in mixed form

\[
\begin{align*}
-2\mu \nabla \cdot \varepsilon(u) + \nabla p &= f \quad \text{in } \Omega \\
- \nabla \cdot u &= \frac{1}{\lambda} p \quad \text{in } \Omega
\end{align*}
\]

- Definition of stresses

\[
\begin{align*}
\sigma &:= \nu \nabla u - p I \\
\sigma_n &:= \sigma n \\
\varepsilon(v) &:= \frac{1}{2} \left( \nabla v + \nabla^T v \right) \\
\sigma^{\text{sym}} &:= 2\mu \varepsilon(u) - p I \\
\sigma^{\text{sym}}_n &:= \sigma^{\text{sym}} n
\end{align*}
\]

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Numerical experiments for two dimensional problems

- Two dimensional mixed problems: nearly incompressible elasticity and Stokes equations using hdG and Taylor-Hood discretisations.

- Two-level preconditioners with coarse spaces associated to local generalised eigenvalue problems

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- Two-level (5 eigenvectors)

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Main results for two-level methods

Nearly incompressible elasticity

Taylor-Hood discretisation


hdG discretisation

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Numerical results for three dimensional problems

- two-level preconditioners with coarse spaces associated to local generalised eigenvalue problems.
- three dimensional mixed problems discretised by Taylor-Hood elements.

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Conclusion
Conclusion

- A new discretisation method that takes into account non-standard boundary conditions.
- Stabilisation method that allows the use of the same polynomial degrees for velocity and pressure discrete spaces.
- Efficient domain decomposition preconditioners for Stokes equations associated with the non-standard interface conditions.
- Two-level domain decomposition preconditioners for nearly incompressible elasticity and Stokes problems provide the scalable results.
Future work

- Prove the well-posedness of the continuous TVNF and NVTF boundary value problems.
- Applying the preconditioner to other discretisations of the Stokes and linear elasticity problems.
- Extend this method to the Oseen and Navier-Stokes equations.
- Convergence estimates for the spectrum of the preconditioned operator for the saddle-point problems.